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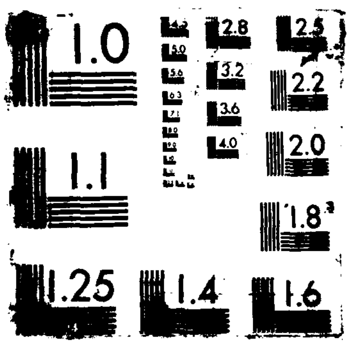
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THE BOUNDED RATIONALITY CONSTRAINT: EXPERIMENTAL AND ANALYTICAL RESULTS

Anne-Claire Alice Louvet

Laboratory for Information and Decision Systems
MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

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THE BOUNDED RATIONALITY CONSTRAINT: EXPERIMENTAL AND ANALYTICAL RESULTS

by

ANNE-CLAIRE ALICE LOUVET

This report is based on the unaltered thesis of Anne-Claire A. Louvet, submitted in partial fulfillment of the requirements for the degree of Master of Science in Operations Research at the Massachusetts Institute of Technology. The research was conducted at the MIT Laboratory for Information and Decision Systems with support provided by the Office of Naval Research under contract no. N00014-85-K-0329 and by the Basic Research Group, Technical Panel on C^3, Joint Directors of Laboratories under ONR contract no. N00014-85-0782.

Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA 02139

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Signature of Author ____________________
Interdepartmental Program in Operations Research
May 13, 1988

Certified by Alexander Levis ____________________
Dr. Alexander H. Levis
Thesis Supervisor

Accepted by ____________________
Prof. Amedeo R. Odoni
Graduate Registration Officer, Operations Research Center

THE BOUNDED RATIONALITY CONSTRAINT: EXPERIMENTAL AND ANALYTICAL RESULTS

by

Anne-Claire A. Louvet

Submitted to the Department of Electrical Engineering and Computer Science on May 13, 1988 in partial fulfillment of the requirements for the degree of Master of Science in Operations Research.

ABSTRACT

The bounded rationality constraint sets an upper limit on the rate with which decisionmakers can process information satisfactorily. This rate is studied both experimentally and analytically. A simple computer game for a single decisionmaker was used in which subjects were asked to find the smallest of a set of ratios present on the screen for a limited amount of time. Both the amount of time (twelve values) and the number of ratios (two values) were varied A Gompertz curve is used to model the experimental results and establish the existence of a time threshold beyond which performance decreased significantly. An information theoretic model of the cognitive workload is used to estimate the workload associated with the tasks. The time threshold T* and the cognitive workload lead to a value for the bounded rationality constraint for each subject and each number of ratios. The distribution of the bounded rationality constraint across subjects for each number of ratios is found to be normal. Also, the bounded rationality of each subject as the number of ratios is changed does not vary significantly. These results may be used in the design of multi-person experiments and eventually in the methodology for organization design. First, a single value of the bounded rationality constraint for each decisionmaker would be needed for similar types of tasks. Second, since the distribution of the bounded rationality constraint across subjects appears to be normal, establishing the threshold level for a sample of decisionmakers could be sufficient to estimate the level for a larger population.

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Title: Senior Research Scientist

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CHAPTER I

INTRODUCTION

1.1 OVERVIEW

Performance of human decisionmakers under time constraints is a critical measure in information processing and decisionmaking organizations, especially when the time constraint is very severe as for example in tactical military organizations. One of the major determinants of performance of individuals under time constraint is their ability to increase their rate of processing as the rate of information input to the system increases. The hypothesis is that human decisionmakers may not increase their processing rate indefinitely. March (1978), developed the idea that decisionmakers are limited by the "cognitive capabilities of human beings," and introduced the concept of bounded rationality. It is assumed that as the rate of input information increases, subjects reach a critical rate of information processing after which performance decreases drastically in an unpredictable manner. This rate, identified as the bounded rationality constraint , has been related to the cognitive workload associated with the different tasks to be performed as well as the input rate of information using information theory (Levis,1984).

1.2 STATEMENT OF THE PROBLEM

No experimental work has been done to study the bounded rationality constraint of human decisionmakers. The purpose of this thesis is to confirm experimentally the existence of a maximum information processing rate and investigate its stability both across tasks and across people. Because the bounded rationality constraint is defined as an information processing rate, two critical values are involved: the task input interarrival rate and the amount of information processing required to perform the task. Therefore, both experimental results and analytical results are required. First, an experiment was designed and run under the direction of Dr. Jeff T. Casey at the Laboratory for Information and Decision Sytems. The results were analyzed to estimate the maximum task interarrival rate before the decisionmakers' performance decreased significantly. Then, the amount of information processing required to perform the task of the experiment was computed using N-dimensional information theory.

The hypotheses which are posed are the following. First, the bounded rationality constraint exists and may be identified experimentally by a sharp decrease in performance. Second, when considering similar tasks, the maximum processing rate should be stable within an individual. Finally, it is hypothesized that the bounded rationality constraint is reasonably stable across well trained subjects.

1.3 THE THESIS IN OUTLINE

The second chapter of this thesis, Chapter II, presents an overview of workload and the general analytical models that are used to analyze and model the experiment. The experimental procedures are described and explained in Chapter III. Chapter IV analyzes the obtained results and the first conclusion is drawn to affirm the existence of the bounded rationality constraint. Postulations are made about the stability of the bounded rationality constraint both across similar tasks and across decisionmakers. Chapter V describes the different algorithms that were chosen as models of the subjects' decision process, and attests the models' plausibility by comparing the performance of the algorithms' simulations and the subjects' performance. In Chapter VI, the methodology and assumptions used to compute the workload of each algorithm are detailed, and the numerical values are evaluated for each algorithm. Finally in Chapter VII, the bounded rationality constraint is derived for each subject for two different tasks and the hypotheses are tested. The bounded rationality does not only exist, but is both stable across similar tasks and across subjects.

CHAPTER II

THE ANALYTICAL TOOLS

2.1 WORKLOAD

The analytical framework, used for modeling the simplified air-defense tasks presented in this thesis, is that of n-dimensional information theory. It is build upon two primary quantities: entropy and transmission. Entropy is the fundamental measure of information and uncertainty : given a variable x, an element of the alphabet X, occurring with probability p(x), the entropy of x, H(x), is defined as follows:

$$H(x) \equiv - \sum_{x} p(x) \log p(x) \tag{2.1}$$

Entropy is defined in bits when the base of the logarithm is two. Entropy is also defined as the average information or uncertainty of x, where information does not refer to the content of the variable x, but rather to the average amount by which the knowledge of x reduces the uncertainty about it.

Transmisison T(x:y) is also known as mutual information. The transmission between variables x and y, elements of the alphabets X and Y, given p(x), p(y), and p(x|y) (the conditional probability of x, given the value of y), is defined as follows:

$$T(x:y) \equiv H(x) - H_y(x) \tag{2.2}$$

where

$$H_y(x) \equiv - \sum_{y} p(y) \sum_{x} p(x|y) \log p(x|y) \tag{2.3}$$

is the conditional uncertainty in the variable x, given full knowledge of the value of the variable y. Transmission may be interpreted as the amount by which knowledge of x reduces the uncertainty in y, or vice versa, as it is a symmetric quantity in x and y.

McGill (1954) generalized this basic two-variable input-output theory to N dimensions by extending Eq. (2.2):

$$T(x_1:x_2:\ldots:x_N) = \sum H(x_i) - H(x_1,x_2,\ldots,x_N) \tag{2.4}$$

The N-dimensional mutual information measures the total constraint or interrelatedness holding among all N variables of a system.

The workload surrogate, denoted by G, is defined as being the total processing activity of the system, i.e., the sum of the entropy of all the variables in the system.

$$G = \sum_{i=1}^{N} H(w_i) \tag{2.5}$$

Using the Partition Law of Information, noted PLI (Conant, 1976), the total activity G may be decomposed into components that characterize how information may be transformed as it is processed by a system. For a system with N-1 internal variables, w_1 through w_{N-1}, and an output variable, y, also called w_N, the law states

$$\sum_{i=1}^{N} H(w_i) = T(x:y) + T_y(x:w_1,w_2,\ldots,w_{N-1}) + T(w_1:w_2:\ldots:w_{N-1}:y)$$

$$+ H_x(w_1,w_2,\ldots,w_{n-1},y) \tag{2.6}$$

and is easily derived using information theoretic identities.

The left-hand side of Eq. (2.6) represents the total activity, G, of the system. The other terms of equation may be interpreted in the following way. The first term, T(x:y), is called throughput and is designated G_t. It measures the amount by which the output of the system is related to the input. The second quantity, $T_y(x:w_1,w_2,\ldots,w_{N-1})$,

$$T_y(x:w_1,w_2,\ldots,w_{N-1}) = T(x:w_1,w_2,\ldots,w_{N-1},y) - T(x:y) \tag{2.7}$$

is called blockage and is designated G_b. Blockage may be thought of as the amount of information in the input to the system that is not included in the outpout. The third term, $T(w_1:w_2:...:w_{N-1}:y)$ is called coordination and is designated G_c. It is the N-dimensional transmission of the system, i.e., the amount by which all of the internal variables in the system constrain each other. The last term, $H_x(w_1,w_2,...,w_{N-1},y)$, designated by G_n, represents the uncertainty that remains in the system variables when the input is completely known. Although this information is called 'noise' since it originates within the system, it is not necessarily adverse, as the word usually connotes; the decisionmaker may introduce information previously held to ease the decision process. The partition law may be abbreviated:

$$G = G_t + G_b + G_c + G_n \tag{2.8}$$

2.2 THE DECISION-MAKING MODEL

2.2.1 Overview of the Model

Whereas in classical decision theory the decisionmaking organization has an unlimited amount of time in making the decision, in tactical battle situations, time pressure is one of the most critical features of the decisionmaking process. In the first case, the decisionmaking organization has a good knowledge of all the actions that may be taken, and reasonable estimation of the consequences or costs of each action. In the latter case, as mentioned in the decision-analysis literature, the ability of the decisionmaking organization to analyze and process the input messages, formulate actions, and foresee consequences is limited.

2.2.2 The Decision-Making Organization Model

The basic model of the memoryless decisionmaker with bounded rationality is a two-stage process illustrated in Figure 3.1.(Boettcher, 1981). Specifically, it is assumed that the two stages are (a) situation assessment (SA), and (b) response selection (RS). The decisionmaker recieves an input symbol x_i from the environment with average interarrival time τ . The Situation Assessment stage (SA) of the decision process contains algorithms that process the incoming signals x_i to obtain the assessed situation z. It consists of a set of U algorithms (deterministic or not) that are capable of producing some situation assessment z of the set Z. The choice of algorithms is achieved through specification of the internal variable u in accordance with the situation assessment strategy p(u), or p(u|x), if a decision aid (e.g., a preprocessor) is present. The RS stage contains algorithms that produce outputs y_j of the set Y in response to the situation assessment z and the command inputs. The selection of the algorithm is made according to the response selection strategy p(v|z). The two strategies, when taken together, constitute the internal decision strategy of the decisionmaker. The structure of this model has been extended to include interactions with other organization members, as well as memory, but the extended model goes beyond the scope of this thesis. The assumptions under which the model was used in this work are first that the model is memoryless (memory was investigated by Hall, 1982, and Bejjani, 1985), second, there is no preprocessor (decision aids and preprocessors were studied by Chyen, 1984).

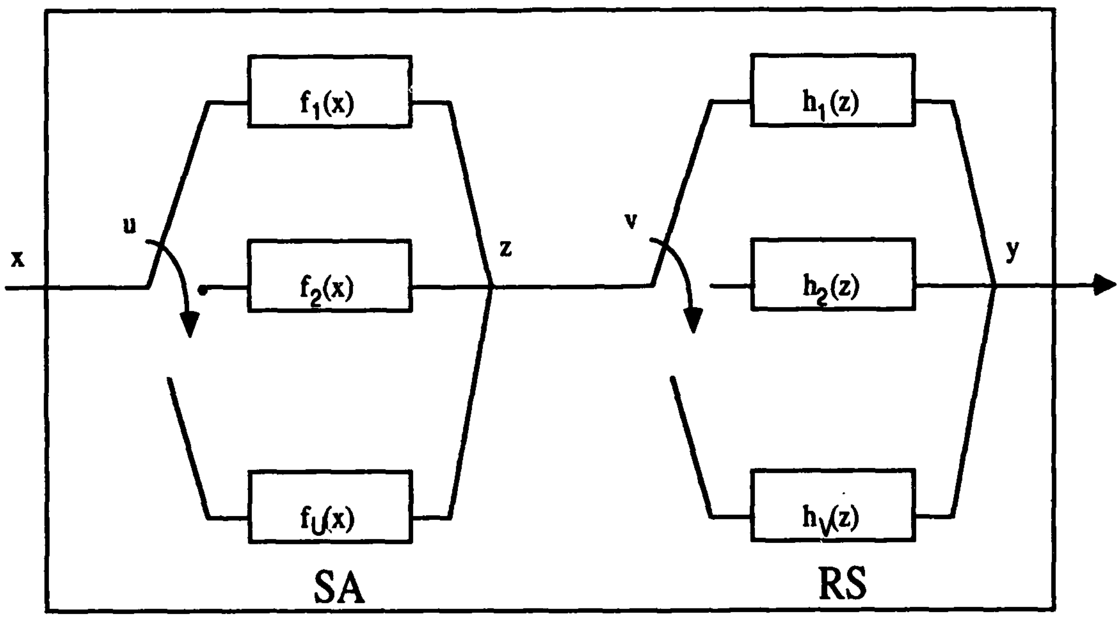


Figure 2.1 Two-Stage Decisionmaking Model

2.2.3 The Task Model

It is assumed that the DM receives signals $\underline{x}_i$ ε X from the source with interarrival time τ. These inputs are assumed to take values from a finite set called the input alphabet and noted X. The cardinal of the set X is noted n. Each element $\underline{x}_i$ of the set X, has a probability p_i of being emitted and is assumed to be statistically independent of the other inputs. Also the set X is exhaustive, that is:

$$\sum_i p_i = 1 \qquad (2.9)$$

The decisionmaker's task is defined as processing the input symbols x_i to produce output symbols y_j of a finite set Y. Such a task implies that the organization designer knows a priori the set of desired responses Y and, furthermore, has a function or table L(x) that associates a desired response or a set of desired responses y_j, elements of Y, to each input x_i of X. (This implication is used later on in this work to estimate the performance of the subjects and is therefore of interest.)

2.3 THE BOUNDED RATIONALITY CONSTRAINT

The first chapter of this thesis introduced the assumption that the processing rate of human decisionmakers is bounded. The concept of bounded rationality constraint has been studied both in experimental psychology and in the domain of C^2.

2.3.1 Experimental Psychology

In the experimental psychology and behavioral analysis literature, one may find two different approaches which may be related to the concept of human bounded rationality: the Yerkes-Dodson 'law' and decisionmaking under time pressure. (Casey, 1987 a, c).

Considerable experimental psychological work has examined the influence of arousal on performance in various types of tasks. Figure 2.2 shows the relationship between arousal and performance called the Yerkes-Dodson 'law'. This relation is shown when arousal is varied over an extremely wide range. Arousal is influenced by a variety of factors including cognitive workload. At very low arousal, performance is low due to boredom and vigilance limitations. At very high arousal, performance is also low, but it is then due to stress and sensory overload. In a well designed organization, all decisionmakers should be operating near the top of the curve. (Casey et al. , 1987).

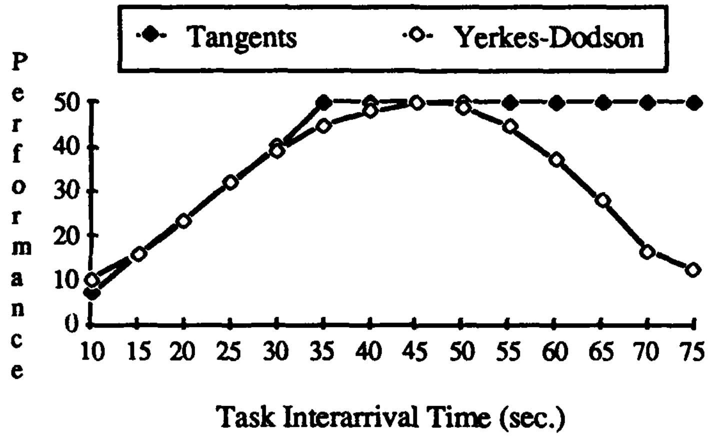


Figure 2.2 : The Yerkes- Dodson Law.

Decisonmaking under time pressure, however, has been given very little attention . A few studies in the behavioral decision litterature have been made (Ben et al.,1981; Wright, 1974; Wright et al., 1977). The general conclusion of these works is that when under time pressure, people process only a portion of the information that they would normally process. Further, they filter the process so that the information that is processed is more important than that which is not processed. Such conclusions may have a significant impact when modeling tasks performed under time pressure. In Chapter V of this thesis, theses conclusions are used as assumptions when modeling the task of the experiment which was performed under time pressure.

2.3.2 The C^2 Approach

Time pressure is one of the most significant features of decision-making in the context of tactical battle management, and significant research has been made in the domain of C^2 to study the effect of short action time on the decisionmaking process and on performance, i.e., Cothier (1984) . The concept of bounded rationality in communication theory will be presented before that of information theory to contrast the two different approaches.

In communication theory, the concept of channel capacity defines the maximum transmission between input and output that the particular channel can provide. This constraint alone is not adequate to describe the limitations of a decisionmaker. Some decision tasks such as decision processes with binary outcome may require a lot of internal processing, but very little input-output transmission.

In the information theoretic model, it is assumed that simple information processing tasks are performed with little error when both the rate of information processing imposed by the input interarrival rate is low and the decisionmaker is not bored. As the input interarrival rate increases, the decisionmaker increases his information processing rate. If the information rate increases further still, a point is reached when the decisionmaker may not increase their processing rate anymore: the decisionmaker is overloaded and his performance decreases significantly. The degradation of performance and the decisionmaker's coping strategies are not statistically predictable and may take many forms. Examples of coping strategies may be ignoring entire inputs, simplifying the algorithms used to give less accurate responses, etc..(Miller, 1969)

The notion that the rationality of a human decisionmaker is bounded has been modeled as a constraint on the total activity G (Levis, 1984). The specific form for the constraint for the memoryless and deterministic model has been suggested by the empirical relation

$$t = c_1 + c_2 G_t \tag{2.10}$$

where t is the average reaction time, i.e., the time between the arrival of the input and the generation of an output y, and G_t is the throughput rate computed using the Partition Law of Information (see equation 2.8). It is assumed that the decisionmaker must process his inputs at a rate that is at least equal to the rate with which inputs arrive. The latter has been modeled by τ, the mean symbol interarrival time:

$$t = c_1 + c_2 G_t \leq \tau \tag{2.11}$$

The modeling assumptions in this work are that

$$c_1 / c_2 = G_b + G_n + G_c \tag{2.12}$$

and that c_2 does not depend on p(x). Then, the bounded rationality constraint takes the form

$$G = G_t + G_b + G_n + G_c \leq \frac{1}{c_2}\tau = F\tau \quad (2.13)$$

where F can be considered as a rate of total activity and is measured in bits per second.

Equation 2.13 may be rewritten using the DM's average processing time t as

$$\frac{t}{c_2} \leq F\tau \quad (2.14)$$

For values of τ sufficiently small, noted τ_{min}, the time t required to process the task with acceptable accuracy will equal the lapse of time between two inputs, and inequality 2.13 will become an equality described as :

$$G = F_{max}\tau_{min} \quad (2.15)$$

where

$$t_{per\ input} = \tau_{min} \quad (2.16)$$

and F_{max} is assumed to be the maximum information processing rate, and t the minimum time required to perform the task with the desired accuracy.

The bounded rationality constraint assumes that if the processing rate F_{max} is exceeded, performance will drop significantly in an unpredictable manner. Equation 2.15 may be rewritten as:

$$F_{max} = G / t_{per\ input} \quad (2.17)$$

where the different quantities have already been described above.

From equations 2.15 and 2.17, it is apparent that for the purpose of investigating the behaviour of the bounded rationality constraint, the maximum information processing rate is a function of three different parameters: the total activity required to perform the task, noted G, the input signal interarrival time, noted τ, and the minimum time required to process the information and perform the task with the desired level of accuracy, noted t. These conclusions have a significant impact when considering the design of experiments which will be described in the next chapter.

CHAPTER III

EXPERIMENTAL PROCEDURES

The existence and the behaviour of the bounded rationality constraint are tested and analyzed using experimental results. This chapter describes the experimental setup and procedures which were designed and run under the direction of Dr. Jeff T. Casey at the Laboratory for Information and Decision Systems. First, the relevant parameters are characterized in section 3.1. Then, the experimental procedures are reviewed in section 3.2. Finally, the purpose of the task constraints and experimental setup are explained in sections 3.3 and 3.4.

3.1 THE PARAMETER TO MANIPULATE

The information processing rate F is described in Chapter 2 as being a mathematical function of three different parameters, the cognitive workload required to perform the task, the minimum time required to perform the task for a given level of accuracy, and the input signal interarrival time. (See equations 2.15 and 2.17). When considering the maximum processing rate noted F_{max}, these three parameters may be reduced to two, since the assumption is that when F_{max} is reached, the input interarrival rate is equal to the minimum processing rate. As a result, the parameter "time" may be considered as the time allotted to perform the task, also called the window of opportunity. Therefore, two different approaches may be used to study F_{max}. One may manipulate either the time allotted to perform the task (t), or the cognitive workload (G) while keeping the other parameter constant.

The effect of the bounded rationality on performance as a function of workload or time allotted per trial has been described as a step function (see Figures 3.1 and 3.2). Performance is stable until the maximum amount of information processing is reached. Then performance drops at or under chance level. The step function represents an instantaneous decrease in performance. It is assumed however, that human decisionmakers will not behave in such a rigid way; when F_{max} is reached, performance will decrease significantly but more smoothly than the step function.

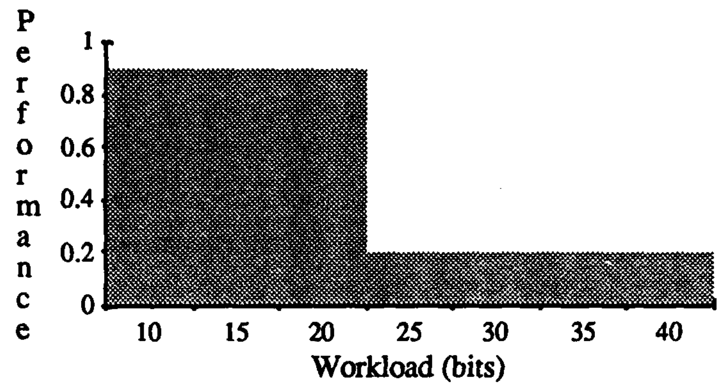


Figure 3.1 Performance as a Function of Workload

The first approach consists of varying the amount of cognitive workload while keeping the time allotted to perform the tasks constant. For a given t, the critical cognitive workload G* associated to F_{max} is measured experimentally as the workload after which performance decreases significantly. The second approach consists of varying the time allotted to perform the task while keeping the workload constant. The methodology is the same as for the first case, but instead of using multiple tasks only one task is used and the time allotted to perform the task is varied. For a given task, the critical time t* associated with F_{max} is measured experimentally. The total activity G, associated with the task is computed analytically using information theory.

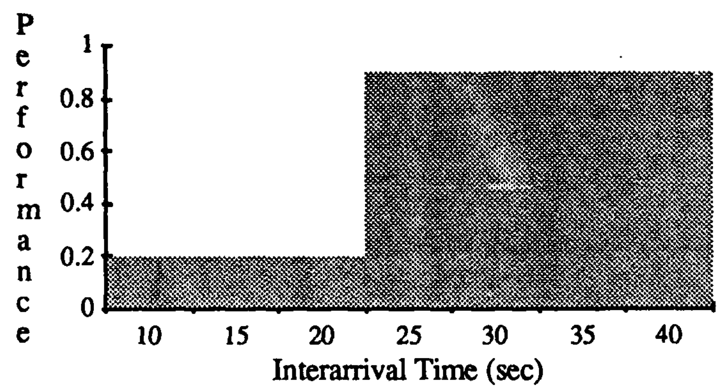


Figure 3.2 Performance as a Function of Interarrival Time

The manipulation of the task processing time is simpler to monitor and control under experimental conditions than the manipulation of workload. In particular, time is a continuous variable whereas the workload is not continuous and needs to be assessed analytically. Therefore, the time allotted per trial is the parameter which was varied.

3.2 EXPERIMENTAL PROCEDURE

This work is only the first in a series of experiments, therefore the simplest decision making organization was simulated: the organization was reduced to a single decision maker. Since little was known about the experimental study of bounded rationality, the task was set so that the factors which were affecting the subjects' performance could be monitored as precisely as possible. The task was also chosen so that the subjects could become 'well trained experts' with reasonable amount of training, thereby satisfying the requirement that the decisionmakers' performance did not benefit from the learning effect during the experiment. The experiment was designed to satisfy both the goals and constraints which were just mentioned and may therefore seem very basic.

This section first describes the experimental setup, then the manipulation of parameters, the organization of trials, the practice sessions and finally the subjects are characterized.

3.2.1 Description of the Setup

The experiment consisted of a highly simplified tactical air defense task . It was run on a Compaq Deskpro Model 2 equipped with an 8087 math coprocessor, monochrome graphics card (640 X 200 pixels), 640K of memory, and monochrome monitor. The experiment was programmed in Turbo Pascal version 3.01A. The operating system was MS-DOS version 2.11. It was also run on an IBM PC AT with the 80287 math coprocessor and with 640K of memory. None of the high resolution graphics capabilities of the AT were used so that the experiment be portable to a wide variety of PC compatible machines.

The computer screen shown in Figure 3.3 consists of three different parts: A large circle, a small circle and a rectangular box. The large circle represents a radar screen. The small circle represents the clock which shows the time allotted for the trial as well as the amount of time left to perform the task. The rectangular box left of the screen and full of 'domino ' shaped rectangles, shows the number of ratios used for the given trial, (4 in this example) and the number of ratios still to be processed (2 in this case). The keyboard was used to enter the subjects' responses.

The experiment consisted of blocks of trials. A trial consisted of either four or seven threats that were to be processed by the decisionmaker within the allotted time shown by the clock. Within each block of trials, the number of ratios was constant and the time allotted per trial was varied in alternating descending and ascending order. Each block of trials was seperated by a longer pause and flashing to indicate that the number of ratios was changing.

For each threat two pieces of information were presented as a ratio of two two-digit integers: relative speed and relative distance from the center of the screen. The distance was in the numerator and the speed in the denominator. Therefore, each ratio represented the time it would take the ratio to reach the center of the screen. The subject's task was to select the ratio which would arrive first at the center of the circle in the absence of interception. The task can be interpreted as one of selecting the minimum ratio.

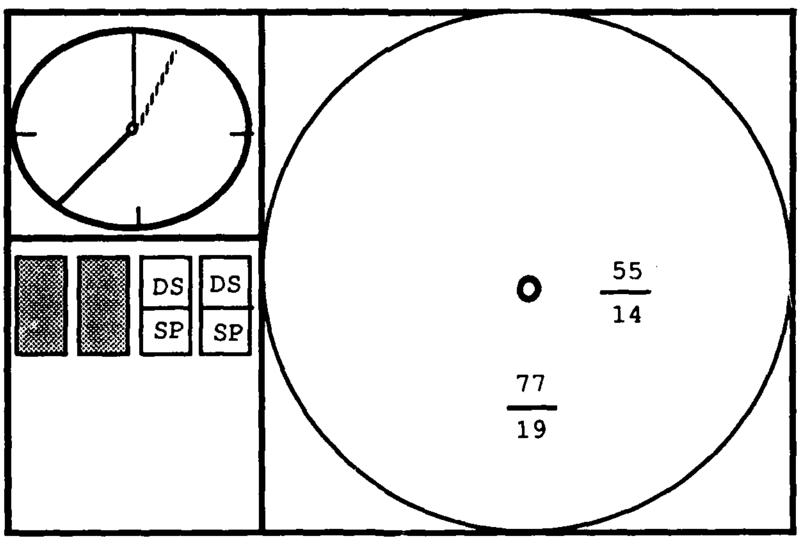


Figure 3.3 The Screen Display Used in the Experiment

For each trial, only two ratios were identifiable and present on the radar screen at the same time. The other ratios were shown on the side of the screen by the 'domino' shaped rectangles. Such a procedure forced the DM's to process ratios in pairs.

The ratios appeared only on the vertical or horizontal diameter of the radar screen, and the physical distance of each ratio from the center was proportional to the distance of the ratio as indicated by the numerator. Thus ratios appeared in one of four regions: left, right, above, or below the center. Each ratio was randomly assigned to one of these four regions, subject to the constraint that no two ratios appeared in the same region at the same time. For each pair of ratios in a given trial, the subject indicated his or her choice by pressing one of four arrow keys corresponding to the direction of the ratio from the radar screen's center. The ratio chosen as smallest was retained on the radar screen, the other vanished, and the next ratio to be processed was taken from the small rectangle's area and placed on the radar screen. This procedure was repeated until all ratios of the trial had been examined. Row(s) of small rectangles to the left of the radar screen indicated the total number of ratios for the current trial and the number yet to be examined (see Figure 3.3). Each time a new ratio appeared on the radar screen, one of the rectangles turned grey and the numbers within that rectangle disappeared. The subject could not give a final answer until all the ratios had been examined, (three comparisons for four ratios and six for seven). The arrow keys were located on the numeric keypad of the keyboard and were arranged isomorphically with the four regions of the radar screen.

Performance feedback was provided at the end of the trial. When a trial was finished on time, only one ratio remained on the screen at the end of the trial. If this ratio was in fact the smallest, it "flashed" several times to indicate a correct response. If this ratio was not the smallest, a low-pitched tone signalled the error. This tone (which subjects reported to be particularly obnoxious) was used to discourage subjects to use guessing as a strategy. When a trial was not finished on time, the screen vanished so the subject knew he had not answered within the allotted time.

3.2.2 Manipulation of Task Interarrival Time

In usual information theoretic setups, it is assumed that the inputs are emitted by one or many source(s) at a mean symbol interarrival time noted τ. In this experiment, to test the existence of the bounded rationality constraint, the average interarrival time is not held constant, but is varied. However for easier control of the experimental parameters, the time allotted to perform the task (noted t) is monitored, not the interarrival time.

The amount of time allotted for each trial was shown by the fixed clock hand (see Figure 3.3). A moving second hand (running clockwise from 12 o'clock) indicated elapsed time within a trial. A 1.5 second pause prior to the start of each trial allowed subjects to see how much time was allotted. The fixed hand flashed during this interval. Time allotted per trial was varied in alternating descending and ascending series.

One of the questions which were to be answered by this experiment related to the stability of F_{max} across tasks, if it could be shown that F_{max} existed. It was decided to choose two different numbers of ratios to investigate this issue. Therefore one of the questions was

$$\frac{G(4)}{t^*(4)} \overset{?}{=} \frac{G(7)}{t^*(7)} \qquad (3.1)$$

This issue raised another question: When considering the measurements of time allotted per trial, should the time allotted per trial be considered or should the average time allotted per comparison for each trial be considered?

One of the hypotheses was that because of the task setup which only allowed the subjects to consider two ratios at the same time, the cognitive workload required to process the four ratios was approximately twice that required to process trials of seven ratios. In one case three comparisons were required whereas in the other six comparisons were required, and it was assumed that the same algorithmic structure was repeated for each comparison. Equation 3.2 decomposes the workload for one comparison into the internal variables, whereas equation 3.3 shows it for two comparisons.

$$G_1 = H(x_1) + \sum_{i=1}^{k} H(w_i) + H(y_1) \qquad (3.2)$$

$$G_2 = H(x_2) + \sum_{i=1}^{k} H(w_i) + \sum_{i=k+1}^{2k+1} H(w_i) + H(y_2) \qquad (3.3)$$

where x_1 is the input variable and y_1 the output variable for one comparison, and x_2 is the input variable and y_2 the output variable for two comparisons, and there are k internal variables noted w_i for each comparison.

Assuming that the workload per comparison was approximately the same for four and for seven ratios, if it were proved experimentally that the minimum average time allotted per comparison was not significantly different for four and for seven ratios, then F_{max} for both numbers of ratios should be assumed to be not significantly different. Therefore, it was decided that the parameter which should be monitored was the average time allotted per comparison which will be noted T, rather than the time allotted per trial which was noted t. T may be expressed as a function of the number of comparisons m within a given trial as follows:

$$T = t/m = t/n-1 \tag{3.4}$$

where n is the number of ratios.

To study the variations between trials of three and trials of six comparisons, the average time per comparison was set to be the same for both types of trials. (Assuming F_{max} exists, the time threshold associated with F_{max} would be derived from the experimental results, and noted T^*_3 for three comparisons and T^*_6 for six.)

The experiment was also constructed to minimize the influence on performance of time required for non-cognitive (i.e., perceptual and motor) activity. A trial consisted of a set of either three or six comparisons. For a set of three comparisons, the time allotted per trial, noted t, ranged from 2.25 to 10.5 seconds. For a set of six comparisons, t ranged from 4.5 to 21 seconds . Thus the average time per comparison, noted T, was varied from 0.75 to 3.5 seconds in 0.25 seconds increments for both conditions and 12 different values of T were recorded. Since even the minimum average time per comparison of 0.75 seconds allowed ample time for eye movements, perception, and motor response, it could be assumed that the major limiting factor on the performance of the subjects was the bounded rationality constraint F_{max}.

3.2.3 Organization of Trials

The experiment consisted of blocks of twenty four trials within which the number of ratios was kept constant. A block of trials consisted of a descending series over the 12 values of t, followed by an ascending series. Such an alternation between ascending and descending series were aimed at smoothing out the anchoring effect of either only going from minimum to maximum or only going from maximum to minimum. After a block was over, the number of ratios was changed for the subsequent block. There was a 2.5 sec. pause between blocks, during which time, the large rectangle to the left of the radar screen (see Figure 3.3) flashed to indicate the impending change in number of ratios. The pause was aimed not only at showing to the subjects what the next number of ratios would be, but also at allowing to bring the tension down a little.

For each subject, the full experiment consisted of eight blocks of trials for both numbers of comparisons. The number of comparisons changed at the end of each block. (It was considered that the small differences between the difficulty of different trials were to even out when considering blocks of twenty four trials). The subject's response was recorded and mapped with the expected solution. Immediate feedback showed the subject whether the answer was correct or not. Such a method satisfied the subject's curiosity about the accuracy of his previous decision. It also allowed the experimenter to estimate the subject's overall performance and ability to cope with time pressure.

The goal was to study the subjects' degradation of performance, therefore it was important to make sure that the range of time intervals for which the subjects were tested was large enough so that both a stable performance and a degradation of performance could be observed. The subjects had to be tested both over time intervals that were large enough so that their performance was close to optimum, and also small enough so that their performance be below chance level.

By observing the subject run one session of the experiment, it could often be estimated if the experiment was well calibrated for the particular subject, i.e., if the time window used to test the subject was well chosen. For some of the subjects the experiment was run over larger time intervals because preliminary analysis of their data showed that the time window used was not large enough to gather all the relevant information. Since an inappropriate experimental setup was not always spotted on time, subjects for whom the experiment was not run properly were asked to come for extra sessions. As a result, for some subjects, more data has been collected.

For the subjects who only came for the scheduled sessions, the total duration of the experiment was approximately 2.5 hours, divided in three sessions: eight blocks of twenty-four trials were completed in each session and subjects typically participated in no more than one session per day. To limit fatigue, each session was seperated into four ten-minute subsessions between which the subjects could take a brake. This was to allow them to relax and have good attention span during the short subsessions. Prior to each experimental session, subjects were given a brief (three to five minute) "warmup" period during which no data were recorded.

3.2.4 Practice Session

Subjects received a 30 minute practice session prior to the actual experiment. This session consisted of six blocks of trials over T for each number of ratios. For the practice session, T was varied from 1 to 5 sec. per comparison in 0.5 sec. increments. Informal discussion with subjects indicated that most felt their performance would not improve substantially with additional practice. Practice was important because the subjects were not supposed to improve their performance as the experiment was run; the analytical tools developed by Boettcher et al. assume that the subjects are both well trained and qualified to perform the task. The practice session was also useful in getting some feedback from the subjects. A few subjects decided not to go on with the experiment, whereas some were advised not to participate in the study. The few subjects who were asked not to participate were people who were not familiar at all with approximation or rounding-off procedures necessary for such a task. They could not meet one of the requirements necessary to use information theory when applied to decision making or decisionmaking organizations: well trained and qualified decsionmakers. Except for those few special cases, the practice data were not analyzed.

3.2.5 Subjects

Twenty-five subjects ran the experiment to its full extent, since one subject was eliminated from the sample. Almost three quarters of the subjects (nineteen) were present or former MIT students (both graduates and undergraduates), the others were MIT employees or students' friends. The large number of MIT students is not inappropriate since MIT students should be qualified to perform the task and, as mentioned above, the subjects should satisfy this requirement.

3.3 PURPOSE OF VARYING THE NUMBER OF RATIOS

It was assumed in section 3.2.2 that the amount of workload per comparison was approximately the same for trials of four and seven ratios. However, the effect of manipulating the number of ratios was of some intrinsic interest, because of implications for how subjects manage their time. Effective time management is more critical for seven than for four ratios, while "overhead" or "start-up" time is more critical for four ratios than for seven.

Therefore, if the value of the subjects' threshold (assuming it exists) was not significantly affected by changes in the number of ratios, it could be established that, to some degree, the bounded rationality constraint is stable across tasks. If, however, instability were found for such a minor task change, there would be no need to go further.

Subjects knew before the start of each trial how much time, t, was allocated for the trial. Part of the subject's task was to budget the available time over the three or six comparisons so that all comparisons could be completed and full use made of the available time. The criticality of accurate budgeting can be seen from Equation (3.5).

$$\text{Response Time} = m\,T' + b \tag{3.5}$$

where m is the number of comparisons (three or six), T' is the average amount of time the subject allocates to each comparison, and b is the overhead, startup, or initialization time for a trial. It is assumed that the value of b is independent of m. According to this model, the subject must choose T' so that the resulting response time is less than or equal to t. Clearly, with increasing m, the detrimental effect of setting T' non-optimally increases relative to the detrimental effect of the fixed overhead, b.

3.4 PURPOSE OF THE TASK CONSTRAINTS

3.4.1 Constraints on the Experimental Setup

In order to constraint the strategies the subjects could use, two restrictions (already mentioned in section 3.3) were imposed. First, ratios were displayed in pairs and only one pair was identifiable at a time . Second, a final response was permitted only after all of the four or seven ratios had been displayed. These two procedures forced the subjects to make a given number of comparisons -three when four ratios and six when seven- or at least forced them to consider all the ratios. Having a more precise idea of the steps the subjects went through is an essential tool when computing the workload, since workload is dependent on the amount of information that the subjects process. Such restrictions also eliminated the variation in the order of information acquisition which could increase the workload, if the subjects had been hesitant when deciding which ratios to consider first.

Within the rest of the thesis, since one of the goals is to study the difference between trials of three tasks and trials of six tasks, a trial will be defined as a set of three or six tasks, where one task corresponds to finiding the smallest of two ratios.

3.4.2 Instruction to the Subjects

Subjects were instructed to attend only to the numeric information of each ratio even though the physical distance of each ratio from the center was proportional to its numeric distance (Casey, 1987 b). This was done to restrict the number of strategies the subjects would use.

This restriction is important, because Greitzer and Hershman (1984) showed that an experienced Air Intercept Controller tended to use physical distance information only (and not speed information) in determining which of a number of incoming ratios to prosecute first. This simplified strategy was labeled the *range* strategy. The operator was, however, able to use both range and speed information -- the *threat* strategy -- when instructed explicitly to do so. The threat strategy, if executed in a timely way, is of course more effective than the simpler range strategy.

3.4.3 Constraints on the Ratios

Another method, which was used to monitor as closely as possible the amount of work the subjects did, was to impose constraints on the ratios. The ratios were very carefully chosen to equalize the difficulty of the different comparisons and trials. (Changes in performance were not to be caused by differences in task difficulty, but because of overload.)

For each trial, all ratios were either greater than or less than one. This restriction was included because pilot work had shown that decisions involving ratios on opposite sides of one were trivially easy, regardless of interarrival times. The greater-than-one / less-than-one determination was made randomly for each trial.

Speeds and distances were selected subject to the following constraints:

(1) greater than 10 and less than 98,
(2) no multiples of 10.
(3) Each speed and distance combination was screened and rejected if the resulting ratio was a whole number,

Additional constraints were that :

(4) no speed value be used more than once per trial;
(5) no distance value be used more than once per trial;
(6) no speed value be the same as its corresponding distance value; and
(7) no two ratios have the same value.

Distances were selected independently of speeds, but had to satisfy constraints six and seven.

The second round of pilot experiments included these constraints. The subjects, however, reported that some comparisons were still much easier than others. It appeared that the ratios less than one could be very difficult to compare because the numerical values could be very close. To avoid especially difficult comparisons, new constraints were imposed on trials of ratios less than one. For the same reason, the ratios larger than one were also constrained. As a result, the candidate ratios obtained applying all the constraints mentioned above were screened against the following new criteria:

(8) each possible pair of ratios within a trial of ratios less than one must differ by no less than 0.05 and by no more than 0.9 and;
(9) in the greater than one condition, the minimum allowable ratio was 1.2;

If a candidate ratio failed on any criterion, a new ratio was generated and the process was repeated until a complete set of four or seven compatible ratios had been obtained. (An attempt was made to impose the same constraints on both the ratios less than and larger than one, but when doing so, it was sometimes impossible to generate seven ratios larger than one satisfying the appropriate constraints.)

3.5 FEEDBACK FROM THE SUBJECTS

Generally, subjects seemed to be challenged by the experiment. Many subjects reported that the experiment forced them to concentrate hard and that they were glad that each session was seperated into subsessions between which they could relax. Also, it was a common feeling that there was a breakpoint after which they could not process the task within the required time anymore. A few subjects mentioned that they had had a harder time with trials consisting of ratios larger than one than with ratios less than one. Such a difference was not built in purposely, but is described and explained in Chapter V: the algorithms which were used by the subjects resulted in a higher performance for ratios larger than one than for the ones less than one. Also, some subjects reported having a difficult time with the keyboard: the response that they had chosen was not always the response that they entered through the keyboard. (Most of the subjects made at least one error just because they had just hit the wrong key!) Such errors will be one of the sources of noise and discrepancies which are found in the data. Finally, it appeared that there was a delay between the instant when the key was pushed and the answer was recorded. This delay was particularly critical for the small values of T, since subjects tended to answer as late as possible; sometimes their right answer was not recorded.

CHAPTER IV

THE EXPERIMENTAL RESULTS

In Chapter III, the experimental setup was described. Chapter IV analyzes the experimental results with respect to the hypotheses that may be tested experimentally. First, in section 4.1, the data recorded during the experiment is presented and the hypotheses are stated. In section 4.2, the methodology used to test the different hypotheses is described. In section 4.3, the procedures required prior to testing the hypotheses are presented. In section 4.4, the data is analyzed according to the different procedures and, in section 4.5, conclusions are drawn from the experimental results.

4.1 THE DATA AND THE HYPOTHESES

4.1.1 The Data Collected

This section first describes the data recorded in general terms. Then, two examples are given to explain how to reconstruct the data from the recorded data files.

For each trial, seven different data sets were recorded. (See Table 4.1) First the average time allotted per task is shown in column 1. The average time varied between 0.75 sec. to 3.5 sec. for most subjects. The number of ratios for the trial is shown in column 2: either four or seven ratios, i.e., three or six tasks. In column 3 is noted whether the time per trial was increasing or decreasing: 1 indicates a descending series whereas 2 indicates an ascending series. The subjects' performance is recorded in column 4. The subjects received a score of 0 if an answer was given but it did not match the correct answer, a score of 2 if no answer was given within the allotted time, and finally a score of 1 if the answer matched the correct one. Column 5 lists the two digit distance, followed by the two digit speed of each ratio in the order they appeared on the radar screen. In column 6 are inscribed the ratio number that the subject chose at the end of each comparison. Finally in column 7, the time (in 10^{-2} seconds) the subject used to process each task is noted.

As an example of how to read the data files, two rows of Table 4.1, (noted *1 and *2 in the table), are described. The trial recorded in the row, *1, may be described as follows. The average time T per task was 3.00 seconds, and there were four ratios, (three tasks), in

Table 4.1 Sample of the Data Collected: Subject 50, Session 1, First Set of Three Tasks

	Col. 1 Time T	Col. 2 # of Ratios	Col. 3 Asc./ Desc.	Col. 4 Perf. J	Col. 5 Speed and Distance of the Ratios	Col. 6 Result of Comparison			Col. 7 Elapsed Time to Completion of Task #		
						1	2	3	1	2	3
	3.50	4	1	1	2686316766873891	1	1	1	204	99	127
	3.25	4	1	1	7344513949248857	2	2	2	214	308	290
*1	3.00	4	1	1	4364185844521563	2	2	4	181	110	165
	2.75	4	1	2	5919652537139531	2	3	3	368	247	220
	2.50	4	1	1	8297298431424676	2	2	2	241	71	82
	2.25	4	1	1	1289368253656283	1	1	1	132	77	55
	2.00	4	1	1	4652118619514157	2	2	2	104	104	49
	1.75	4	1	2	3764111562971634	1	1	0	373	161	0
	1.50	4	1	1	3161179212425881	2	2	2	176	66	38
	1.25	4	1	2	5716822144129622	1	0	0	395	0	0
	1.00	4	1	2	2769347114634358	1	0	0	296	0	0
	0.75	4	1	2	7139763588657537	1	0	0	242	0	0
*2	0.75	4	2	2	6245934837228267	1	1	0	192	11	0
	1.00	4	2	2	3192218148724351	1	0	0	302	0	0
	1.25	4	2	2	6947743525166452	1	1	0	302	82	0
	1.50	4	2	2	7596488753865563	2	2	0	201	230	0
	1.75	4	2	1	1452139539692939	2	2	2	182	55	44
	2.00	4	2	1	2555146124311798	2	2	4	181	104	151
	2.25	4	2	1	5369164165752785	2	2	4	307	127	137
	2.50	4	2	1	2233269464752959	2	2	2	187	99	105
	2.75	4	2	1	4383647834393763	1	1	1	242	131	104
	3.00	4	2	0	5691135651926887	1	3	3	126	225	132
	3.25	4	2	1	9862685588489673	2	2	2	263	121	263
	3.50	4	2	1	2779596614213681	1	1	1	159	94	258

this trial (as indicated by the 4 in column 2). Then, the 1 in column 3, indicates that this trial is part of the descending series: the T value was larger before this trial. The 1 in column 4 indicates that at the end of the trial, the subject had correctly chosen the smallest of the four ratios. From column 5, the value of each ratio for this particular trial may be read. The four different ratios were:

$$R_1 = 43 / 64 \qquad R_2 = 18 / 58 \qquad R_3 = 44 / 52 \qquad R_4 = 15 / 63$$

From columns 6 and 7, the last information may be derived. Subject # 50 used 1.81 seconds (column 7, first number) to decide which was the smallest ratio of the first task: The ratio # 2 was chosen, (see column 6, first digit). Then, between the result of the first task and that of the second, 1.10 seconds had elapsed (see column 7, second number), and the subject had chosen ratio 2, (see column 6, 2nd digit). Finally, it took the subject 1.27 seconds to compare the last two ratios (ratios 2 and 4), and enter the final solution, ratio 4.

The trial recorded in the row, *2, may be described as follows. There were four ratios, (three tasks), and the average time per task was 0.75 seconds. This trial was during an ascending series (a 2 in column 3), and the subject did not answer in time, (indicated by a 2 in column 4). The values of the four ratios were as follows, (see column 5):

$$R_1 = 62 / 45 \qquad R_2 = 93 / 48 \qquad R_3 = 37 / 22 \qquad R_4 = 82 / 67$$

Finally, the subject chose ratio 1 as the smallest of ratios 1 and 2 after 1.92 sec. and ratio 1 again as the smallest of ratios 1 and 3 after 0.11 sec. The subject then ran out of time before entering a final solution.

4.1.2 The Hypotheses

The hypotheses which were to be tested using the experimental results were the following:

Hypothesis(1): Decisonmakers are subject to the bounded rationality constraint, that is the bounded rationality constraint sets an upper limit on the amount of information that decisionmakers can process before their performance decreases drastically.

Hypothesis(2): If the bounded rationality constraint exists, assuming that the workload for six tasks is approximately twice that for three tasks, (see section 4.2), is there a significant difference between the value of the bounded rationality for three tasks and that for six tasks for each subject?

In Chapter VII, two more hypotheses are tested combining the experimental and analytical results. The first is designed to confirm that F_{max} is stable for each subject as the number of tasks is varied. The second tests the stability of F_{max} across subjects.

4.2 THE PROCEDURES TO TEST THE HYPOTHESES

4.2.1 The Existence of the Bounded Rationality Constraint

This section first describes the tests necessary to prove the existence of the bounded rationality constraint. Then, the theoretical model, 'single step ', and the empirical model, growth curve, are discussed. Finally, the growth curve is characterized.

In section 3.1, the theoretical model associated with the existence of the bounded rationality constraint is described as a 'single step' function. Such a model is not feasible when considering concrete applications; humans do not behave in such a rigid and structured way, and unwanted noise always distorts experimental results. The empirical model which will be used to prove the existence of the bounded rationality constraint is a growth model (described in the next paragraph). The first hypothesis, the existence of the bounded rationality constraint, may be restated in terms of growth curves as follows:

(1) a growth model fits the data well;
(2) a growth model will fit the data better than a linear model;
(3) the existence of a time threshold (which will be noted T*), may be identified and constructed from the growth curve model. This threshold corresponds to the corner point of the step function shown in the theoretical model Figure 3.2.

The existence of F_{max} will be proved first by showing that the growth curve is a good model of the data, (same general characteristics and large R^2). The second step will be to show that a growth curve fits the data better than a straight line, i.e., it is possible to identify a time threshold (breakpoint) after which performance decreases significantly. This will be done by showing that R^2, the coefficient of multiple determination, is consistently larger for a growth curve fit than for a linear fit. (In a third step, the time threshold T* is evaluated for each subject in section 4.4.3)

The following paragraphs describe the general attributes of the family of growth curves. These curves are characterized by an S shape: the growth starts slowly (characterized by a nearly flat curve segment), then the growth increases rapidly (steep slope) and finally levels off. A growth curve seems most appropriate to describe the experimental data, since it characterizes patterns where quantities increase from near zero to close to the maximum level very rapidly.

For the purpose of this experiment, the most appropriate curve of the family of S curves is the Gompertzt curve which has the characteristic of not being symmetric about the inflection point. This is a relevant property, since one can not predict that performance will decrease in a symmetric way after the subject is working beyond the bounded rationality constraint.

The Gompertz curve has three degrees of freedom and is given by (Martino, 1972):

$$J(t) = a\, e^{-be^{-ct}} \qquad (4.1)$$

where J is performance expressed as a value between 0 and 1.

The Gompertz curve may be characterized the following way:

The asymptotes are:

$$\text{At } t = 0,\quad J(0) = a\, e^{-b} \qquad (4.2)$$

$$\lim_{t \to \infty} J(t) = a \qquad (4.3)$$

The inflection point occurs at :

$$t_{inf} = \ln(b) / c \tag{4.4}$$

and the value of J at the inflection point is:

$$J_{inf} = a / e^1 \tag{4.5}$$

For linear regression using the least squares method, the Gompertz function may be linearized as follows:

$$Y = A X + B \tag{4.6}$$

where

$$Y = Ln(Ln(a/J)) \tag{4.7}$$

$$X = t \tag{4.8}$$

$$A = -c B \tag{4.9}$$

$$B = Ln(b) \tag{4.10}$$

4.2.2 Stability of F_{max} Across Similar Tasks

When considering the experimental results, the stability of F_{max} may be studied assuming that the workload for six tasks is approximately twice that for three tasks. (See section 3.3) Therefore, in this chapter, the stability of F_{max} is tested only with respect to T*, the time threshold (introduced in sections 3.1 and 4.2.1). T* is assessed for each subject for both three and six tasks in section 4.5, after the existence of the bounded rationality constraint has been proved. Then, the distribution over subjects of T* for three and six tasks is evaluated separately, and the type of each distribution is compared. Finally, the significance of the difference between the mean of the T^*_3 and T^*_6

distributions are compared using a statistical test, the t test. The hypothesis is validated, if the statistical tests conclude that the two distributions are of the same type and the means are not significantly different. (A 0.95 level of confidence is used.)

4.3 THE PROCEDURES PRIOR TO TESTING THE HYPOTHESES

4.3.1 The Data Analyzed

Since the hypotheses focused on the subjects' performance, only the data strictly related to the subjects' performance: the time alloted per trial, the number of ratios for the given trial, and the score for the given trial are analyzed. (The rest of the data could provide basic data for future research.)

When assessing performance, a wrong answer and an incomplete answer were treated similarily. As a result, for subject i, for each trial k corresponding to the average time T_j, the score was assumed to be an independant Bernoulli variable with probability p_{ij}.

$$X_{ijk} = \begin{cases} 1 & \text{If the tasks were completed within the alloted time and the correct ratio was chosen.} \\ 0 & \text{Otherwise.} \end{cases} \qquad (4.11)$$

An estimate of p_{ij}, was computed as follows using the simple unbiased estimator $\bar{p}_{ij}$

$$\bar{p}_{ij} = \sum_{k=1}^{24} X_{ijk} \; / \, N_O \qquad (4.12)$$

where N_O is the number of times the subject was run for each time interval. For most subjects N_O is equal to 24.

The estimated performance was plotted against the average time allotted per task for Subject #23 in Figure 3.1 and in Figure 4.2 for Subject # 35.

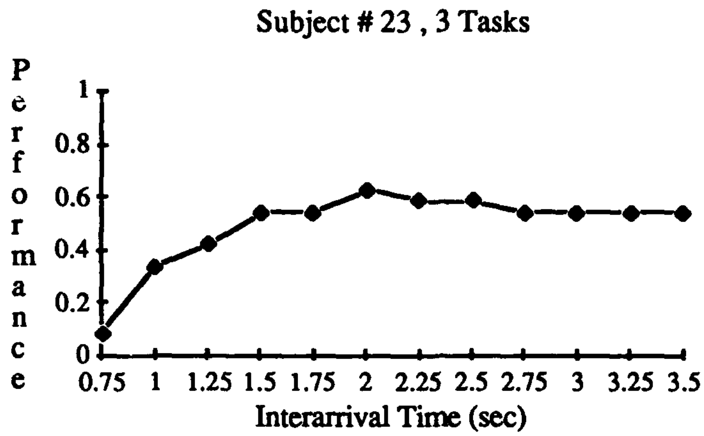


Figure 4.1 Performance Versus Average Allotted Time: Three Tasks, Subject # 23

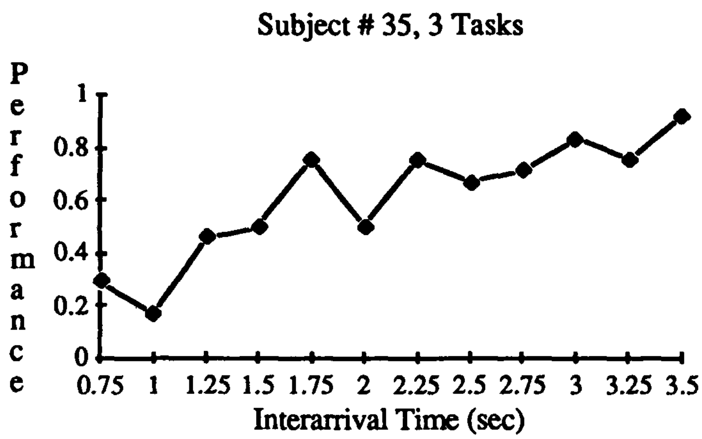


Figure 4.2 Performance Versus Average Allotted Time: Three Tasks, Subject # 35

4.3.2 Data Transformation

Curve fitting will be used to test whether the Gompertz model fits the data well. Since each p_{ij} is the sum of N_O independent identically distibuted Bernoulli variables divided by N_O, each p_{ij} has a different error variance, and one of the necessary assumptions for regression and curve fitting, i.e., equal error variances, is violated.

$$\text{Variance }(p_{ij}) = p_{ij} * (1- p_{ij}) / N_O \tag{4.13}$$

Therefore, in order to equate the error variances, the estimates p_{ij} were transformed using the arcsine formula (see equation 4.14). The denominator ($\pi/2$) is a scaling constant to keep the range of the estimates between 0 and 1; the variances remain equal. The arcsine transformation was used instead of the logit transformation because the logit transformation is more appropriate for data which is symmetric about an inflection point.

$$(\sin^{-1}(\mathrm{sqrt}(\ p_{ij}\ ))\ ) / 1.57 \tag{4.14}$$

Table 4.2 shows the impact of the arcsine transformation on seven different values ranging between 0 and 1. (Values 1/4, and 1/7 have been chosen since they are the performance which would be expected if the subjects were simply guessing for the trials of three and six tasks respectively.) The general effect of the arcsine transformation is to slightly increase small values, while slightly decreasing large values. Since it has most effect on both the lower and upper values, the arcsine transformation will tend to make a threshold, (if there is any), less visible. The difference between the maximum and minimum performances will be reduced as the whole curve is 'squeezed' and flattened.

Table 4.2 The Effect of the Arcsine Transformation

Value	Transformed Value
0.0	0.0
1/7	0.247
1/4	0.334
0.4	0.436
0.5	0.500
0.6	0.564
0.8	0.705
0.9	0.796
1.0	1.000

All analyses reported herein are based on the transformed estimates which will be called performances.

4.3.3 The Gompertz Curve Regression

A computer package, RS/1, (Bell Labs) was used to estimate the Gompertz curve parameters for each data set, and evaluate the fit, the R^2. The program first asked for the function to use as a curve fit. The Gompertz function was typed in. Then it asked where to find the x values and the y values; these were stored in a table, the same for all subjects. The program then wrote the partial derivative of J with respect to a, b and c, and asked for starting values for a, b and c, as well as a convergeance criterion. The selected starting value for a was different for each subject since the subjects' maximum performance was chosen. The same starting values for b and c were entered for every subject, 2 for b and 1 for c. Choosing different starting values in the same range would not have made any significant difference since for each subject the program ran by iteration until the error converged to 0.0001. When a performance of 0 was encountered the computer transformed it to a small value, apparently in the range of 0.00001.

4.4 APPLICATION OF PROCEDURES AND RESULTS

4.4.1 General Characteristics of the Data Analyzed

Performance versus the average time allotted per task was plotted for each subject for both three and six tasks for the transformed data. The curves appeared to have the following set of characteristics:

(1) They do not have the Yerkes-Dodson concave shape. This indicates that the experiment succeeded in tapping into the moderate-to-high arousal portion of the Yerkes-Dodson curve (see Figure 2.2), rather than the "vigilance" portion.

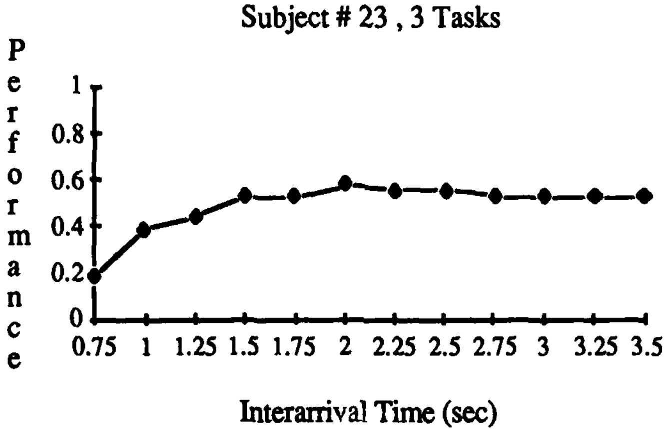
(2) Most curves tend to be flat (zero slope) for large values of T.

(3) They have positive slopes for smaller values of T.

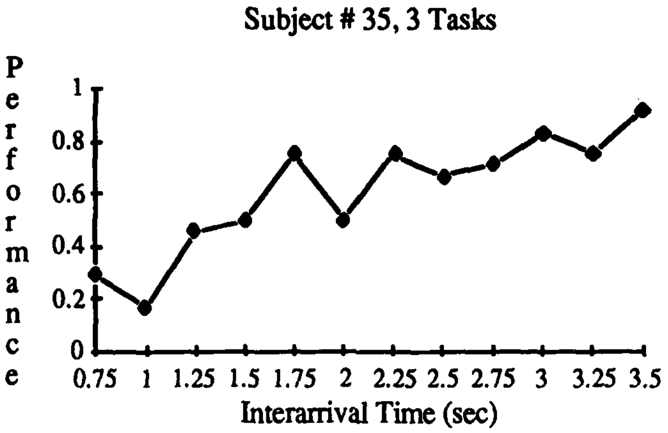
(4) Performance drops and tends to level off for small values of T.

Figure 4.3 shows performance versus the average time allotted per task, t, for two subjects. These curves were selected as being examples of strong, (a), and average, (b), representation of the threshold hypothesis. (These curves are the same as in Figures 4.1 and 4.2, but with the estimated performance.)

Only half of the subjects have have more than one data point below chance level because the allotted time could not be decreased indefinitely. It was necessary that poor performance be caused by mental and not physical limitations. The subject needed enough



(a)



(b)

Fig.4.3 Transformed Performance versus Average Allotted Time per Task for Two Subjects.

time to press a key. One subject was eliminated from the sample, because the experiment was not run properly (inappropriate time window) and the subject was not available for further testing. As a result the population sample was reduced to twenty-five subjects.

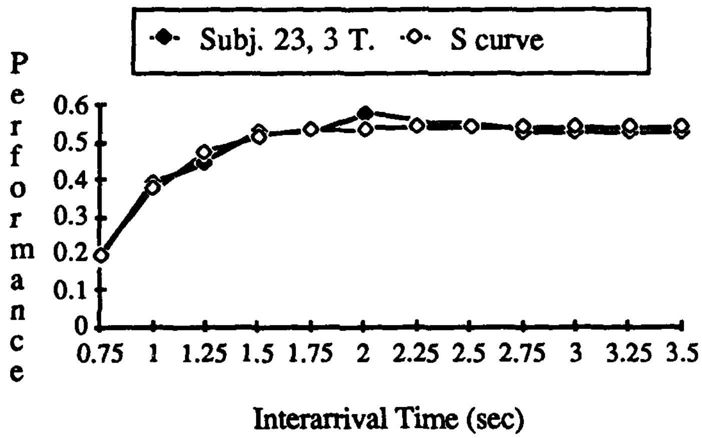
The characteristics of the curves describing subjects' performance as a function of average time allotted per task, suggest that a Gompertz curve could be appropriate for summarizing the data.

4.4.2 The Existence of F_{max} : the Gompertz Fit

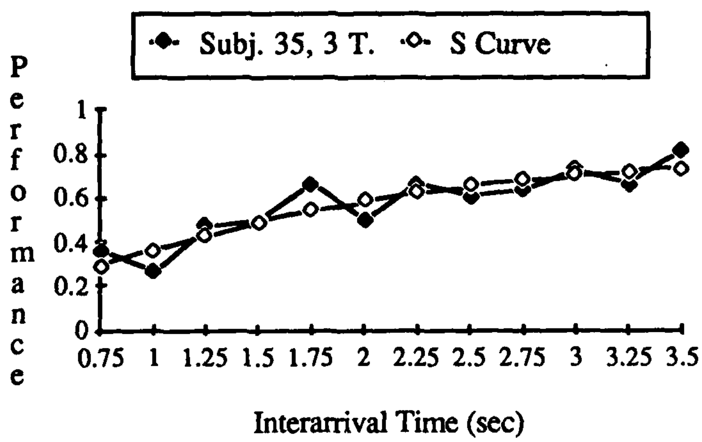
The three parameters a, b, and c of the Gompertz curve were derived for each subject for trials of both three and six tasks and are shown in Appendix A. The parameter 'a' ranged from 0.42 to 0.83, the parameter 'b' ranged from 1.61 to 222.78, and 'c' ranged from 0.77 t o 7.15. The distribution of the values for parameter 'b' was *not* uniform: for trials of three tasks, 23 of the 'b' values were less than 25.00 whereas for trials of six tasks, there were 22 'b' values less than 25.00. The large values taken by the parameter 'b' for some of the subjects was due to the following reasons. First, performance J, is not very sensitive to changes in b. Second, a very small convergence criterion was used in the regression. Finally, by combining equations 4.2 and 4.3, b may be expressed as the logarithm of the ratio of the performance at T equal zero, to the performance as T tends to infinity. Therefore, if the subject's performance for very small T values is 0 or close to 0, b will be very large. In the five cases when the parameter 'b' was exceptionally large, for the lowest T values, the subjetcs' performance was very close to 0.

In every case the Gompertz fit was good: the min R^2 was 0.93, and a check of the residuals showed no consistent pattern which could indicate that the Gompertz was not an appropriate model. Also, in every case, the Gompertz fit was at least as good and almost always significantly better than a straight line fit. (See Appendix B). R^2 ranged from 0.93 to 0.99 for the growth curve, whereas for the linear regression, R^2 varied from 0.45 to 0.93. A one sided statistical t test was made to verify that the R^2 for the Gompertz fit were significantly larger than that for the linear fit. The t value obtained was 23.7. It is much larger than the maximum t* value which would confirm that the two distributions are not significantly different. ($t^*_{0.95,24}$=1.078 for a one sided test with a 0.95 level of confidence and 24 degrees of freedom.). In section 4.4.1, the characteristics of the data were described as being similar to the characteristics of the Gompertz curves. These observations, combined with the large R^2 values for every subject indicate that the Gompertz curves are a good description of the data. The t test confirms the Gompertz' good fit as well as the existence of a time threshold T* (which will be evaluated in section 4.4.2): The bounded rationality constraint exists.

Figure 4.4 show the Gompertz fit superimposed on the observed data. The subjects and the number of ratios are the same than the ones used for Figure 4.3 a-b.



(a)



(b)

Figure 4.4 *The Gompertzt Fit for Two Subjects*

4.4.3 Evaluation of T*

The existence of F_{max} was proved for every subject. Before testing the stability of F_{max} , procedures to evaluate T* are needed. This section describes how T* may be found both analytically and graphically.

In order to stay as close as possible to the theoretical model, (the corner point of the 'single step' function), T* was defined as the point at the intersection of the following tangent lines: the asymptotic performance (the parameter 'a' of the Gompertz curve), and the slope at the inflection point of the Gompertz curve. (See Figure 4.5). The first line forces performance to be at maximum, whereas the other is a good approximation of the speed at which the subject reaches maximum performance as T increases. Had the slope between the maximum and minimum asymptotes been constant, that slope would have been chosen. Figure 4.5 shows the tangent lines and resulting T* value for the same S curve as shown in Figure 4.4 a.

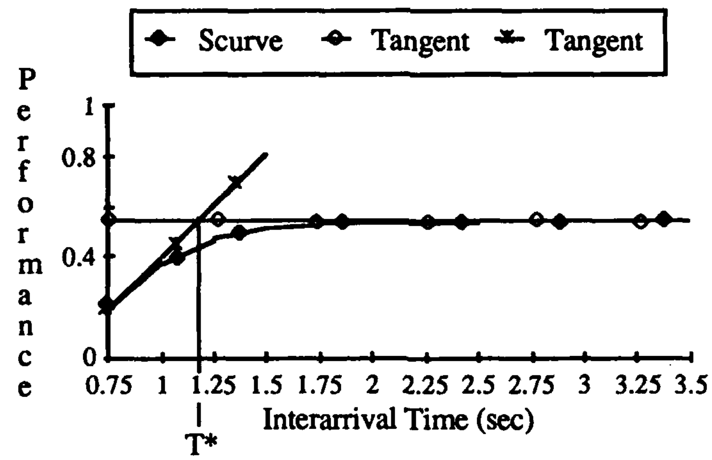


Fig.4.5 Construction of T* using Tangents

Analytically, T* may be also found as the intersection of the two lines:

$$\begin{cases} J = a \\ J = \alpha\ T^* + \beta \end{cases} \tag{4.15}$$

where a is the asymptote of the Gompertz fit.

Therefore:

$$T^* = (a - \beta) / \alpha \tag{4.16}$$

where α is the slope at inflection point and β is intercept of the tangent at the inflexion point.

Since, (See equations 4.4-4.5)

$\alpha = a\, c / e^1$, and $J_{inflexion} = a / e^1$, and $T_{inflexion} = \ln(b) / c$,

then,

$$J_{inflexion} = \alpha T_{inflexion} + \beta \tag{4.17}$$

$$\beta = a\ (1 - \ln (b)) / e^1 \tag{4.18}$$

Substituting α and β in equation 4.16 we obtain the folowing expression for T*

$$T^* = [e^1 - 1 + \ln (b)] / c \tag{4.19}$$

where b and c are two of the three parameters of the Gompertz curve.

It is interesting to noticė that the asymptote of the Gompertz curve, the parameter a, is not present in the equation. The sensitivity of T* with respect to a is nonetheless larger than that with respect ot b or c, since a is related to T* through b and c by the Gompertz model. (Further computations have shown, as expected, that T* is more sensitive to a than it is to b or c.)

4.4.4 The Stability of F_{max} Across Similar Tasks: T^*_3 versus T^*_6

For each subject i, T^*_i was computed for both 3 and 6 tasks and noted $T_i{}^*{}_3$ and $T_i{}^*{}_6$. The obtained T^* values are shown in Appendix C and summarized in Table 4.3. Both the mean value and the standard deviations were very similar for three and six tasks: 2.079 sec. versus 2.069 sec. for the mean and 0.651 sec. versus 0.579 sec. for the standard deviation .

Table 4.3 Summary of T^* Values (sec.) for Three and Six Tasks

	Mean	Std. dev	Min.	Max.
Three Tasks	2.079	0.651	0.911	4.046
Six Tasks	2.069	0.579	1.080	3.504

Generally, the subjects had T^* values for three and six tasks that were very close. A little over half of the subjects, thirteen out of the twenty-five, had a larger T^* for three tasks than for six tasks. Also, since the mean of T^* over subjects were very close for three and six tasks, only a 0.01 difference), one was tempted to conclude that there was no systematic difference in the T^*'s as a function of the number of ratios. To confirm such a hypothesis, a few tests had to be performed. First, one had to check that the two distributions were of the same type, and then, that their mean was not significantly different.

The slightly larger standard deviation of the T_3^* distribution was mostly due to one significantly larger T_3^* value: 4.046 sec. The subject who had a high T_3^* was not performing especially worse for three than for six tasks but the performance was increasing more irregularly. He had complained about the setting of the experiment, and reported entering several times the wrong answer because of inadvertently pressing the wrong key.

A plot of the distribution of the T^*'s for three tasks (Figure 4.6) and for six tasks (Figure 4.7) leads to the hypothesis that the two distributions are normal.

It is interesting to note that in the case of three tasks, most of the difference between the T^* distribution and the normal distribution is due to the fact that the distribution of the T^*'s is extremely peaked. In the case of six tasks, the difference is caused both by the smaller T^* values as well as by the peak around the mean. The Chi-Square test consists of evaluating the difference (noted Q^2) between the distribution under study and (in this

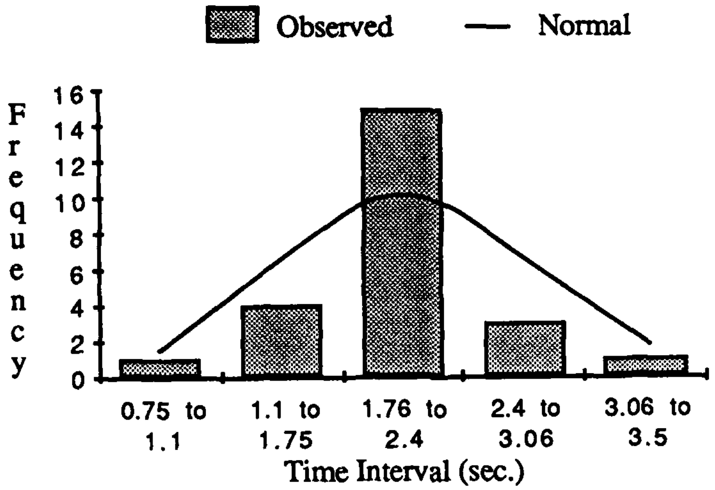


Figure 4.6 Distribution of the T* Values for Three Tasks

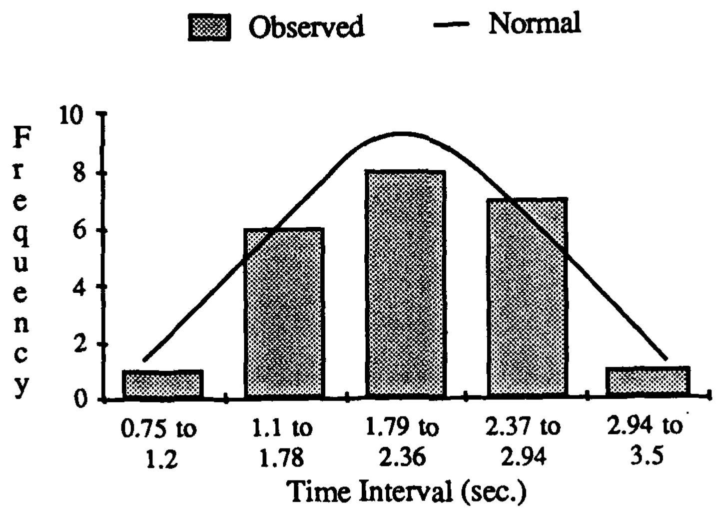


Figure 4.7 Distribution of the T* Values for Six Tasks

case), the normal distribution; Q^2 is computed as follows:

$$Q^2 = \sum_{i=1}^{5} (\text{Observed}_i - \text{Expected}_i)^2 / \text{Expected}_i \tag{4.20}$$

The Q^2 values were 5.6 for three tasks and 4.4 for six tasks which were both smaller than the critical value: $\chi^2_{,0.95,3} = 5.99$. Thus, it could be concluded that the two distributions were both not significantly different from a normal distribution, and were of the same type. (A detailed description of the results of the Chi square tests is given in Appendix D).

The next step was to compare the mean value of the T* distribution for three and for six tasks. A statistical test, the t test, was run. (The test performed is the t test used when comparing two dependent samples. See Appendix D.) The t value obtained was 0.09 ($t = 0.09 < t^*_{23,.95} = 1.74$) which confirms the hypothesis that the two distributions were not significantly different.

Therefore, it may be concluded that T^* is robust to minor task changes, and assuming that the workload for six tasks is approximately twice that for three tasks, the same may be postulated for F_{max}. As a result, each subject i was assigned a single value T_i^* which was equal to the average of $T_i^*{}_3$ and $T_i^*{}_6$. The frequency distribution of the individual T_i^*'s was plotted. (See Figure 4.8). This distribution is unimodal, very peaked, and has mean 2.074 sec. and standard deviation 0.549 sec.

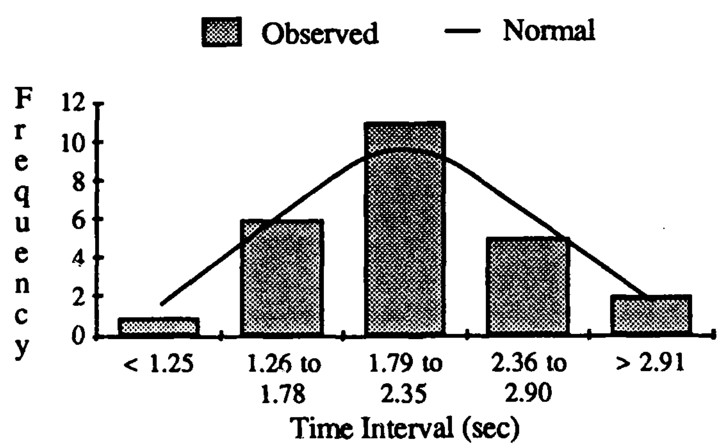


Fig. 4.8 Distribution of the Average T_i^* Values.

The distribution of the T_i^*'s for three and that for six tasks was shown to be normal. Such was also the case for the individual T^* values -- A χ^2 test for goodness of fit revealed non-significant deviation from normality: $Q^2 = 4.4 < \chi^2(.95,2) = 5.99$.

The fact that the T^* distribution is normally distributed is of interest since one may postulate that F_{max} for each subject will also be normally distributed. If this postulation is confirmed in Chapter VII by the analytical results, then the hypothesis that F_{max} is stable across subjects will be validated.

4.5 CONCLUSIONS

The existence of the bounded rationality constraint, F_{max}, has been proved by the experimental results. T^*, the time threshold associated with the bounded rationality constraint, has been evaluated for each subject and both numbers of tasks. It was shown that the T^* value for three and six tasks were not significantly different. Therefore, under the assumption that the workload for six tasks is approximately twice that for three tasks, one may conclude that F_{max} is stable when minor task changes are made. Finally, a T^* value was estimated for each subject. The distribution of the individual T^*'s was normal. Such a result enables the postulation that F_{max} is stable across subjects.

The stability of F_{max} both across similar tasks, and across subjects will be confirmed in Chapter VII when both the experimental and analytical results are combined. First, however, models of the algorithms used by the subjects are presented in Chapter V. Then, in Chapter VI, the workload associated with these algorithms is evaluated.

CHAPTER V

THE DECISIONMAKING MODEL: THE SUBJECTS' VIEWPOINT

5.1 GENERAL PURPOSE

The goal of this thesis is to study the bounded rationality constraint F_{max}. Such a study requires both experimental and analytical results. In Chapter IV, the experimental results were described: the existence of F_{max} was proved, T* was evaluated for each subject, and postulations were made about the stability of F_{max} across tasks. The next goal of this thesis is to present the analytical results, (the computation of workload), and confirm the assumptions raised in Chapter IV concerning the stability of F_{max}. To compute the workload associated with the task, the subjects' mental process must be modeled and then transformed into information-theoretic algorithms. This chapter presents basic models of the subjects' mental process.

A mathematical model attempting to describe the subjects' mental process would be of little significance if it was not validated. Therefore, it seemed appropriate to evaluate the appropriateness of these models. After running the experiment, the subjects were asked to describe the algorithm(s) that they had used while running the experiment; these results are described in section 5.2. The major difificulties encountered when modeling the tasks are described in section 5.3. Then, simple mathematical models which took into account the algorithms described by the subjects were developed and are presented in section 5.4. Each subject was assigned to a particular algorithm. Before analyzing these models and computing the workload associated with each (Chapter VI), the appropriateness of the algorithms is evaluated in section 5.5. The performance of the models is compared to that of the subjects.

5.2 SUBJECTS' STATEMENTS

5.2.1 Correspondence with Cognitive Science

From reading the subjects' description of the algorithms used, as well as their general comments about the experiment, it appeared that the subjects felt under time pressure, and that they had been using coping strategies to perform the task. The task was to compare ratios and find which was the smallest. To ensure 100% performance, a computer program would have processed the task by computing the value of each ratio and then comparing the obtained values. It appeared that the subjects often only processed a portion of the input information that they would normally process, if they had more time or aids (even pen and pencil) to perform the task. Subjects used shortcuts and filtering methods that allowed them to processes the most significant information. An example of such behavior were subjects who systematically ignored the second digits of the two digit values of speed and distance. Such an observation is similar to the conclusions drawn from the few studies of time pressure found in behavioral decision literature.(Wright, 1974)

5.2.2 Retrieving Descriptions of the Model(s) Used

As it was mentioned in the previous section, the subjects were asked to describe the algorithm that they had used to perform the task. Before the subjects' statements were studied, different models that would be plausible descriptions of the algorithms were designed. These models were used as guidelines when the descriptions were too vague.

The first task was to translate the subjects' description into a mathematical model. Whereas some subjects seemed able to analyze very clearly the basic mental processes that they have used, others seemed unable to do so. Phrases like 'When the comparison is not obvious...' appeared more often than expected. A study of the rest of the description often gave some idea of the algorithm (or at least the algorithmic structure) used. Here are a few extracts of some of the subjects' answers:

Extract A:

Step 1: Observe left hand column of multi digit fractions
Step 2: Try to look for 8's or 9's in the second column
Step 3: When digits on the left are the same, decide based on second column digits

Extract B:

For ratios <1 compare numerators if the ratios comparable, otherwise obvious
For ratios >1 if comparable try and reduce otherwise want smaller numerator, greater denominator.

The models were aggregated into a few categories which are discussed in section 5.4.

Translating the subjects' description required a subjective methodology where both intuition and 'common sense' played a very important role. Such modeling methods required an evaluation of each algorithm using some test of appropriateness or some other evaluation method. Such a test, which was alluded to in the first section of this chapter, is described in detail, in section 5.5.

5.2.3 The Stages of the Decision Process

In Chapter II, the decision-making model was described as a two stage process. The first stage, the Situation Assessment stage, allowed the decisionmaker to analyze and assess the situation before making a decision in the response selection stage. At each stage, the subject could choose from a set of algorithms to process the information.

When running the experiment, the subjects seemed to be using only one situation assessment algorithm. The algorithm consisted of looking at the clock and understanding how much time they had to compare the ratios, understanding how many ratios would have to be processed, and finally just looking at the value of the ratios present on the screen. The subjects did not mention these first steps which are the obvious steps that one would follow when faced with such a task.

The response selection algortithm varied from subject to subject. It appeared, however, that most subjects used the same algorithm, whatever the input ratios were. The main factor which seemed to induce a change in algorithms was the time allotted to perform the task. When they could not process the task using the strategy they were most comfortable with or their 'optimum strategy', subjects often switched either to a simpler version of the same algorithmic structure, or to a different structure. The subjects were instructed not to guess unless it was an educated guess, but subjects sometimes just picked one of the two ratios randomly, often hoping that the next comparison would be easier. Changes in strategies due to increase in time pressure were very difficult to monitor since most subjects were not even aware of the change, or if they were, did not report it.

As a result, the models that were derived for each subject, encompass both the Situation Assessment and the Response Selection Stages, but do not take into account the subjects' relationship with the clock. For this particular experiment, the two stage decision model of the single decisionmaker shown in Figure 2.1 may be simplified as in shown in

Figure 5.1.

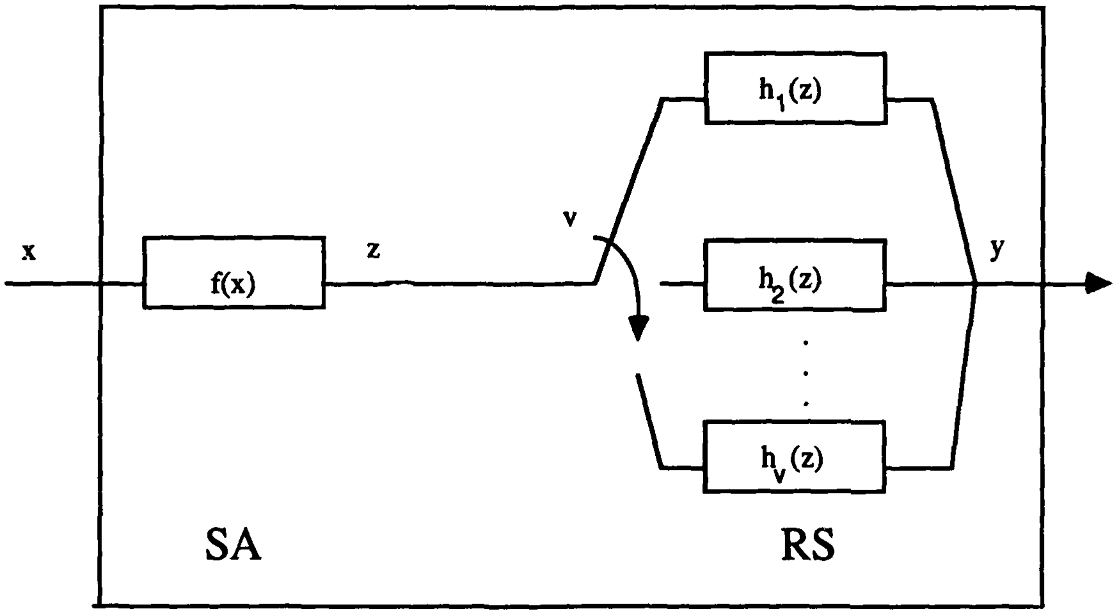


Figure 5.1. The Simplified Decision-Making Model

5.2.4 The Issues of Pure and Mixed Strategies

In the case of this experiment, when considering the type of strategies used by the subjects, the notions of pure and mixed strategies as described by the literature seem difficult to apply. (Boettcher, 1981).

Pure and mixed strategies are defined as follows. In the case of the situation assessment stage, a decisionmaker without a preprocessor uses a pure strategy if whatever the input, he uses a given algorithm to process that input with probability one, (he always uses the same situation assessment algorithm). In the case of the RS stage, the notion is very similar. For each input identified by the situation assessment stage, there is only one response selection algorithm that the DM will use to provide a response. This may be expressed mathematically as follows:

$$p(v = j \mid z = z_i) = 1 \qquad (5.1)$$

where j is the algorithm selected in the response selection

z_i is the output of the situation assessment algorithm.

In the experiment, it was very difficult to evaluate which strategy or algorithm(s) the subjects were using. It was even more so when trying to identify which subject changed algorithm when. Because of the experimental setup, as explained in the previous section, (5.2.3), there was only one situation assessment algorithm, thereby there could only be a pure strategy. For the response selection stage, the setup did not force the subjects to use any particular algorithm. From talking to the subjects and reading their comments, it appeared that the subjects used a single strategy whatever the input was. It is only when they felt too *pressured* that they switched from their 'usual' strategy to a simpler one. Therefore, since the change of strategies was based on one of the input characteristics, (the time available to process the trial), they were using a set of pure strategies for the response selection stage.

5.3 MODELING DIFFICULTIES

5.3.1 Requirements of Information Theory

As described in Chapter II, information theory is a mathematical tool which may be used to compute the cognitive workload associated with a given task. Information theory imposes constraints and requirements on the type of tasks that may be modeled as well as on the algorithms that may be used. These conditions restrict the type of tasks that may be simulated.

One of the major constraints is that the tasks be well defined so that they can be modeled using mathematical variables, or at least variables for which a probability distribution may be derived. As a result, the quantities and parameters which are used must be measurable values, and belong to a finite set.

The other conditions which must be fulfilled are that the decisionmakers be well trained and motivated and that they operate at a level where the bounded rationality is not in effect. The last condition concerning the bounded rationality constraint is particularly important to this section of the research and has serious implications when considering the algorithms that will be modeled to compute the cognitive workload. It has been metioned that subjects have been switching from one algorithm to an other as the time allotted per trial was decreased. When subjects felt overloaded, or close to being overloaded, many switched to an algorithm for which the cognitive workload was less; these algorithms were called coping algorithms. As a result, when modeling the task and assessing the workload, it will be very important to model the algorithm that subjects used when they did not feel under serious pressure yet, i.e., the algorithm that they used when they have the most time available.

The growth curves which were used to model the experimental data smoothed out any change in strategy. Therefore, T* may be considered as an average over several 'T*', each 'T*' associated with an algorithm requiring less cognitive workload: a coping strategy. Since the individual 'T*'s were not identifiable, the T* value (see equation 4.19) was retained. It may also be postulated, that the slope at which performance decreases, (more specifically the slope at the inflection point), reflects the number of different coping algorithms used by the subject as the time available to perform the task decreased: the larger the number of different algorithms used, the smaller the slope, and consequently, the smaller the T* value.

5.3.2 The Limitation of the Mathematical Models

Information theory restricts the type of algorithms that may be used as well as the experimental setups. One of the major problems in trying to assess the mental workload is also derived from the difficulty or better the incapacity to include non-quantitative measures in the mathematical models. How may one model a subject's mental process when the subject describes choosing one ratio over another because 'the comparison was obvious', or how can one describe the fact that another subject will just assume that 2/5 is less than 3/7 ? In both cases, the subject knows (or thinks he knows) the answer and uses some cognitive process to make a decision. No previous research has been done to evaluate and compute using information theory the cognitive workload associated with intuition. The impact of memory on workload has been discussed in the literature (Hall, 1982, Bejjani, 1985). In this research, for simplicity, it is assumed that the decisionmakers are memoryless with respect to short term memory. Also, with respect to long term memory, the only cognitive work which is assessed when choosing the smallest of two single digit ratios is due to the distribution of each ratio. The cognitive work required to retrieve the information from permanent memory is ignored but could be the subject of future research.

5.4 THE RESULTING MODELS

5.4.1 The Different Mental Approaches

When considering all the constraints imposed by the analytical tools as well as by the nature of the task, the number of different approaches was quite small. It appeared that there were only three different basic types of mental processes. Whereas some features were common to all three types, the most important processing in each case was quite different. The three different methods were the following:

Method 1. For each ratio, approximate the speed and distance with single digit values, then compare the resulting ratio.
Method 2. Approximate the ratio (or its inverse) to its nearest integer and compare.
Method 3. Compare the differences between numerators and denominators.

Whereas for the first two methods the first steps could be done independently for each ratio, the last approach included both ratios as soon as some processing was done. Each method resulted in one, two or three different algorithms to include some of the variability among subjects. The resulting set of models consisted of six different algorithms that will be described in detail in the next section. Finally, before performing any computation or approximation, it appeared that the subjects checked for any significantly small ratio. If such a ratio was spotted, they ignored the other ratios and would give the 'small ratio' as the solution. Such a procedure was even more widely spread when the time allotted per comparison was small. For small processing times, the notion of a small ratio was often less strict, and included ratios that would not have been considered if the clock had shown more time available.

5.4.2 The Six Algorithms: Description of the Models.

Models derived from method 1

The first approach (method 1 described above), which consisted of approximating the last digit of both speed and distance, was used by four subjects. Two different algorithms resulted from this approach. The first approximation method, (named Algorithm 1), was to simply truncate the last digit of both speed and distance values when performing the comparison. The second method, (named Algorithm 2), is to truncate first the last digit of the speed and distance values as for Algorithm 1, and then add to the truncated values 0 if the second digit is less than 5 and 1 if the second digit is larger than 5. Once the ratio values have been approximated, the subject has to compare the two resulting ratios. If the two are not equal, the solution is the smallest ratio. If the two are equal, the subject randomly picks one of the two as a solution. Given two input ratios R1 and R2 such that

$$R1=d1/v1 \quad \text{and} \quad R2=d2/v2,$$

one comparison for Algorithm 1 is described in Figure 5.2, whereas one comparison for Algorithm 2 is described in Figure 5.3.

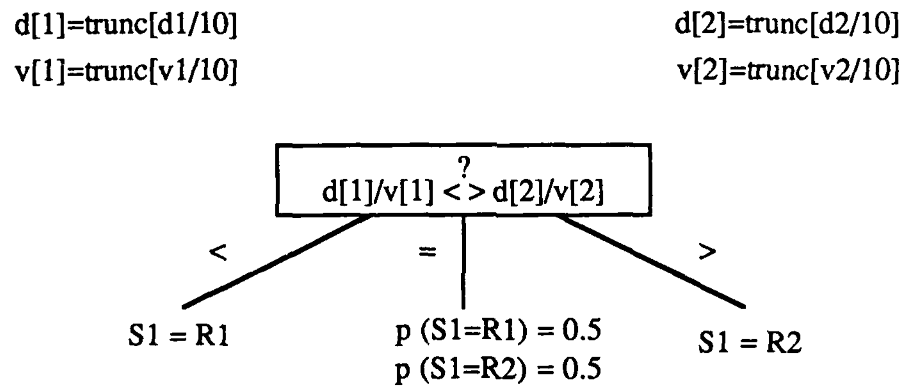


Figure 5.2 One Comparison Using Algorithm 1

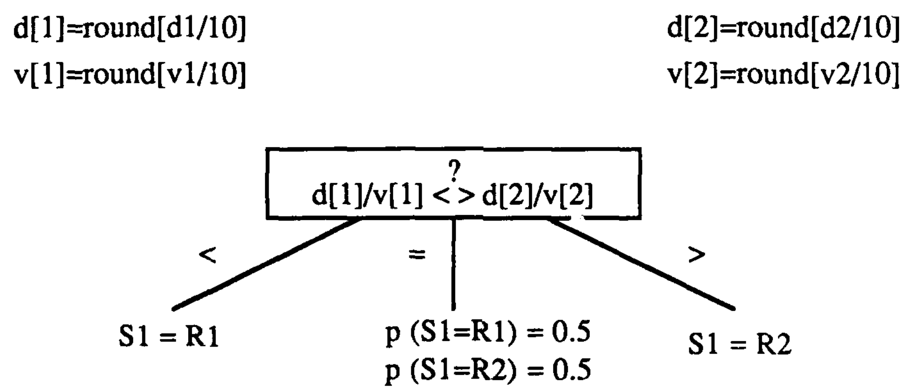


Figure 5.3 One Comparison Using Algorithm 2

Models derived from method 2

Only one algorithm was derived from method 2. This model had the disadvantage of being different for ratios that were less than one and for ratios that were larger than one. For ratios that were larger than one, each ratio was rounded to its nearest integer. Then, if the absolute difference between the nearest integer and the ratio was more than 0.25, the integer value was corrected by positive 0.25 or by negative 0.25, as appropriate. Then the resulting values for both ratios were compared. As for algorithms 1 and 2, if the values were the same, it was assumed that the subjects picked randomly one of the two ratios for the solution. For ratios less than one, the inverse of the ratio is first taken. Then, the same process as for ratios larger than one is used. The resulting algorithm was called Algorithm 3 and the process for one comparison is shown in Figure 5.4 for ratios larger than one and in Figure 5.5 for ratios less than one. Considering the two ratios R1 and R2 already defined for algorithm 1 and 2, Algorithm 3 is described for ratios larger than one in Figure 5. 4 and for ratios less than one in Figure 5.5.

Ratios > 1 $\qquad$ i = 1, 2

rat[i] = round[d_i/v_i]

if [rat[i] - (d_i / v_i)] > 0.25 $\qquad$ then ratio[i] = rat[i] - 0.25

if [rat[i] + (d_i / v_i)] > 0.25 $\qquad$ then ratio[i] = rat[i] + 0.25

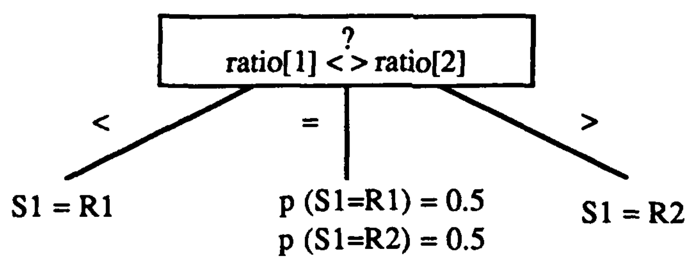


Figure 5.4 One Comparison Using Algorithm 3 for Ratios Larger than One

Models derived from method 3

Three algorithms were derived from method 3 which consisted of comparing the differences between the numerators and denominators (distances and speeds) of the two

Ratios < 1 $\quad i = 1, 2$

rat[i] = round[v_i/d_i]

if (1/ rat[i]) - (d_i / v_i) > 0.25 then ratio[i] = rat[i] - 0.25

if (1/ rat[i]) + (d_i / v_i) > 0.25 then ratio[i] = rat[i] + 0.25

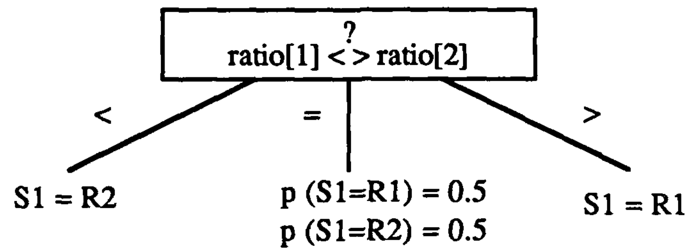


Figure 5.5 One Comparison Using Algorithm 3 for Ratios Less than One

ratios that had to be compared. For Algorithm 4 and Algorithm 5, the difference between the distance and the speed of each ratio was computed, then, the ratio with the smallest difference was chosen. For Algorithm 4, the subject could come to a conclusion if the difference was larger than 10. For Algorithm 5, the subject came to a conclusion if the difference between the speeds was larger than that between the distances or vice versa. The two algorithms are described below in Figures 5.6 and 5.7.

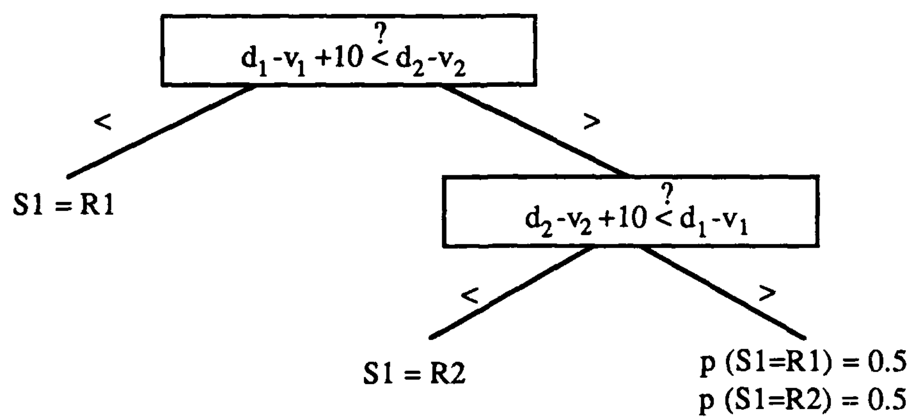


Figure 5.6 One Comparison Using Algorithm 4

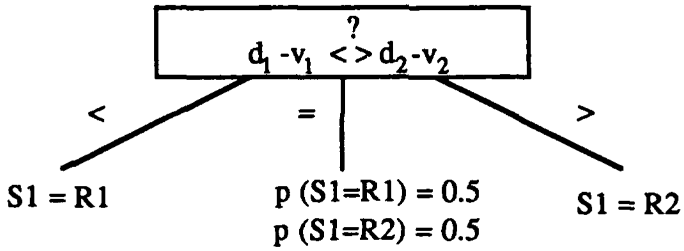


Figure 5.7 One Comparison Using Algorithm 5

The last model, Algorithm 6 is a combination of Algorithm 2 and method 3. The subject first checks if there is not one ratio which has a smaller distance and a larger speed than the other. If he can not make a decision by these criteria, the subject uses the approximation method of Algorithm 2. Algorithm 6 is described in Figure 5.8.

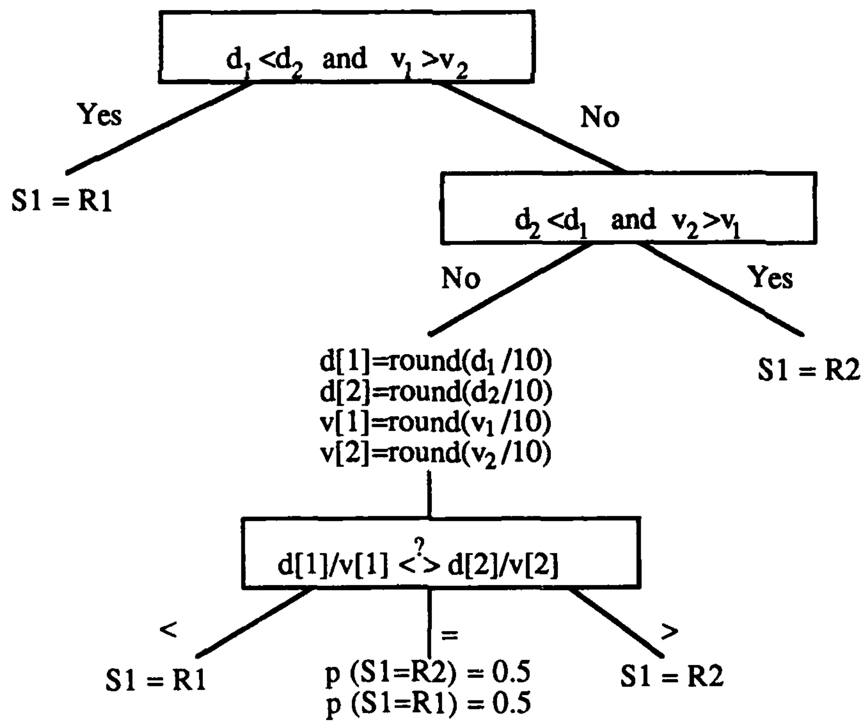


Figure 5.8 One Comparison Using Algorithm 6

5.5 EVALUATING THE MODELS

5.5.1 Purpose of the Evaluation

The different models used by the subjects have just been described. However, before assuming that these models are a reasonable representation of the subjects' mental processes, the appropriateness of these models must be validated. To do so, the maximum performance of each subject will be compared to the estimated performance of the algorithm associated with each subject.

5.5.2 Defining the Maximum Performance

Each subject's maximum performance was established from the experimental results using the S curves. For subject i, the maximum performance is noted a_{i3} for three tasks and a_{i6} for six tasks, and may be derived as follows:

for j = 3 and 6

$$a_{ij} = \lim_{T \to \infty} (J_{ij} (T)) \qquad (5.2)$$

Each of the six algorithms described in section 5.4 represents a pure strategy and is noted f_k, with k taking values ranging from 1 to 6. For a given algorithm f_k, the estimated performance will be noted J_{k3} for three tasks and J_{k6} for six tasks.

The performance that would result from accurately using these algorithms has been estimated by simulating the experiment 300 times on an IBM PC. Each algorithm was programmed in Pascal, and the function "random" was used to generate sets of ratios satisfying the requirements of the experiment, the same way the experiment had been set up. But since whether the sets of ratios were less or larger than one depended on another random function, it seemed important to simulate the experiment for both ratios (larger than and less than one) for the same number of times.

Such a procedure gave particularly relevant information concerning the difficulty of the experiment. Some subjects had mentioned that they found the ratios larger than one more dificult to compare than the ratios less than one. This observation was confirmed by the simulation of the algorithms: the algorithms always performed significantly better for the ratios less than one. Since the trials were independent identically distributed Bernoulli variables, the estimated performance J_{kj} could be computed as follows:

$$\bar{J}_{kj} = (\bar{J}_{kj<1} + \bar{J}_{kj>1}) / 2 \tag{5.3}$$

where:

$$\bar{J}_{kj<1} = \sum_{i=1}^{150} x_{i<1} \ / \ 150 \tag{5.4}$$

$$\bar{J}_{kj>1} = \sum_{i=1}^{150} x_{i>1} \ / \ 150 \tag{5.5}$$

However, since each subject's performance curve had been transformed using the arcsine transformation to perform the regression analysis, it was necessary to make the same transformation on the algorithms' expected performance to have values that could be compared. Therefore, an arcsine transformation was made on the algorithms' simulated performance. Table 5.1 shows the (transformed) ***estimated*** performance for each of the algorithms for three tasks and the non transformed performance both for ratios less than one and ratios larger than one. Table 5.2 shows the same results for six tasks.

The results are only estimates of the population's true mean. The variance for each estimated performance was very low. It varied between 0.0005 to 0.005. (The sample size was 300 of a population of possible combinations of ratios close to 10^{13}.)

The algorithms' estimated performance values were larger for trials of ratios less than one than for trials of ratios larger than one. The difference may be explained by the constraints imposed on the trials. For trials of ratios less than one, the values of the ratios were constrained so that the difference between any two ratios be at least 0.05. The same constraint was not imposed on trials of ratios larger than one for practical reasons: when running trials of six tasks, the program often could not generate ratios satisfying the constraints. Instead, the ratios larger than one were constrained to be larger than 1.2. As a result, the ratios larger than one were on average slightly harder than the ones less than one.

Table 5.1 Estimated Performance for the Six Algortihms for Three Trials

Algorithm number	Estimated Performance (Three Trials) Ratios < 1 untransf.	Ratios >1 untransf.	Overall Perf. untransf.	Overall Perf. arcsine transf.
Al.1	0.84	0.625	0.733	0.654
Al.2	0.86	0.645	0.753	0.665
Al.3	0.91	0.724	0.817	0.719
Al.4	0.744	0.437	0.591	0.558
Al.5	0.757	0.628	0.693	0.627
Al.6	0.86	0.705	0.783	0.692

Table 5.2 Estimated Performance for the Six Algorithms for Six Trials

Algorithm number	Estimated Performance (Six Trials) Ratios < 1 untransf.	Ratios >1 untransf.	Overall Perf. untransf.	Overall Perf. arcsine transf.
Al.1	0.645	0.538	0.592	0.559
Al.2	0.657	0.584	0.621	0.580
Al.3	0.774	0.427	0.601	0.564
Al.4	0.608	0.349	0.479	0.486
Al.5	0.632	0.462	0.547	0.530
Al.6	0.832	0.591	0.711	0.639

Figure 5.9 shows the estimated performance of each algorithm for both three and six tasks. The algorithms perform better for three than for six tasks, but the ordering of the algorithms' performance stays almost unchanged. (Algorithm 3 which performed the best

for three tasks, is only third to best for six tasks. The others have remained unchanged). The average difference between performance for three tasks and performance for six tasks is a 0.1 decrease. Finally, Figure 5.9 also shows that the difference in performance among the algorithms is not very large. For three tasks, there is only a 0.16 difference between the best and the worst algorithm, the difference is 0.15 for six tasks. However, considering the small variances of the algorithms' estimated performance (in the range of 10^{-3}), the differences should not be considered as negligible.

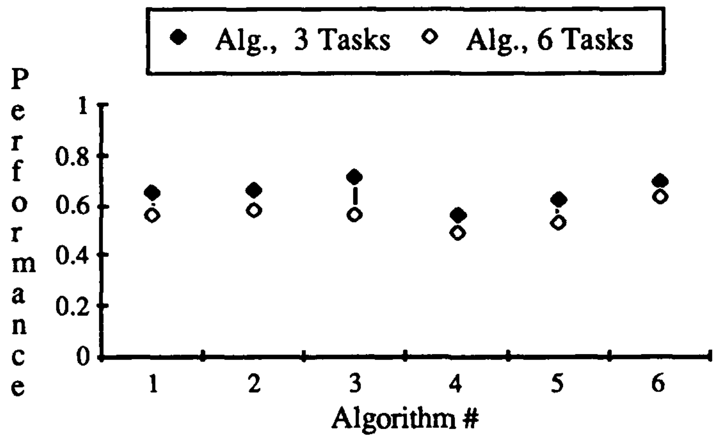


Figure 5.9 The Algorithms' Performances: Three Tasks versus Six Tasks

5.5.3 Comparing Performance: Simulations versus the Experiments

The six algorithms described in section 5.4 were derived from the subjects' descriptions. Each subject was then assigned to the algorithm which was closer to the description he gave. The next step was to estimate the algorithms' maximum performance. The goal of this section is to evaluate the appropriatness of the algorithms.

Table 5.3 shows, for three tasks, the number of subjects who were using each algorithm, the average performance over the subjects and, finally, the algorithm's performance (The subject's performance which was averaged was the asymptotic performance, the 'a' values of the Gompertz fit, see equation 5.2). Table 5.4 shows the results for six tasks. The detailed table, showing each subject's optimum performance

Table 5.3 Three Tasks: Subjects' Performance Versus the Algorithms'

Algorithm #	Number of Subjects Using it	Average Perf. Over the Subjects	Algorithm's Estimated Perf.
1	2	0.573	0.654
2	3	0.590	0.665
3	6	0.715	0.719
4	3	0.555	0.558
5	4	0.655	0.627
6	7	0.682	0.692

for both three and six tasks, as well as the algorithms' performance is shown in Appendix E. The difference between the algorithms' and the subjects' performance was within a close range for three tasks; this is shown explicitly in Figure 5.10.

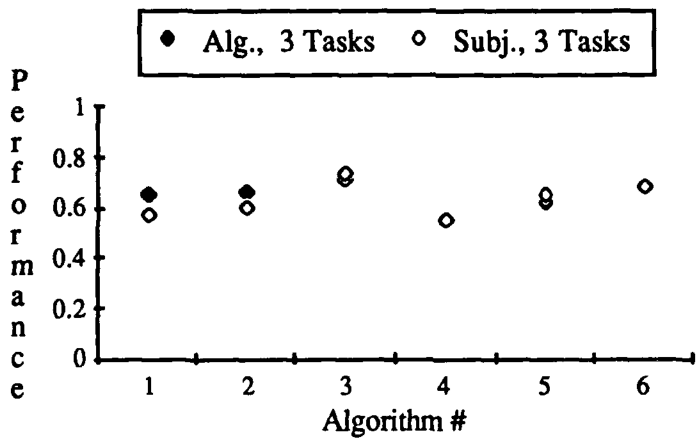


Figure 5.10 The Subjects' Performance Versus the Algorithms': Three Tasks

Three subjects performed significantly better than the algorithms that they seemed to have been using. These subjects were in the School of Engineering and had had very high scores on the SAT's and the GRE's. They seemed very familiar with approximation methods, therefore one may hypothesize that when the algorithms they were using could not give a significant conlusion, they made educated guesses.

For six tasks, Table 5.4 suggests that, on average, the subjects were performing better than the algorithms which were modeled. Since not a single subject mentioned using a different algorithm for three than for six tasks, the algorithms were considered to be satisfactory models.

Table 5.4 Six Tasks: Subjects' Performance Versus the Algorithms'

Algorithm #	Number of Subjects Using it	Average Perf. Over the Subjects	Algorithm's Estimated Perf.
1	2	0.543	0.559
2	3	0.688	0.580
3	6	0.732	0.564
4	3	0.585	0.486
5	4	0.645	0.530
6	7	0.704	0.639

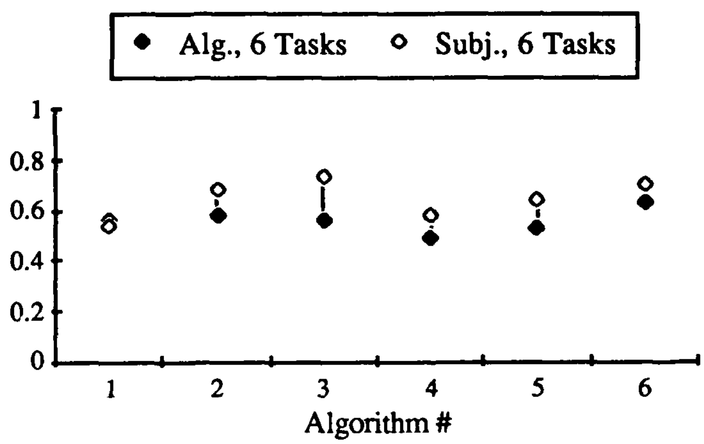


Figure 5.11 The Subjects' Performance Versus the Algorithms': Six Tasks

Overall, the obtained results were satisfactory, and the next step of the research may be described: Computing the workload associated with each algorithm and estimating F_{max} for each subject.

CHAPTER VI

THE WORKLOAD: METHODOLOGY AND EVALUATION

This chapter evaluates the workload for the different algorithms. Each is first transformed into an information theoretic model, i.e., an algorithm for which the entropy of each variable may be computed. Then, the workload for each is evaluated.

Section 6.1 describes the different steps of the modeling process. First, the input alphabet is characterized, but it is impossible to enumerate. Then, the internal variables are reviewed. In particular, the level of detail needed, and the effects of temporary and permanent memory on the assessment of workload are studied. Finally, the impact of having trials of ratios either larger than one or less than one is discussed. Section 6.2 describes the steps followed to compute the entropy of the different variables. Finally , the workload is evaluated in section 6.3. First numerical values of the workload of the different algorithm are given, then the feasibility of these values are discussed and the experimental and analytical results are compared.

6.1 THE INFORMATION-THEORETIC ALGORITHMS

6.1.1 The Input Alphabet

The input alphabet is first defined for both numbers of ratios. Then the size of the alphabets and the input entropies are estimated.

When the subjects start the experiment, the following information is available to them on the computer screen: the number of ratios that are to be processed for the trial, the amount of time they will have to process the task and, finally, the distance and the speed of the two ratios that they will first have to compare. (See Figure 4.1). The time available to perform the task is a parameter which varies from trial to trial.

It is assumed that the amount of cognitive workload required both to acknowledge the amount of time available to perform the task and to register the time available is negligible compared to the workload necessary to process the tasks. Therefore, the input vector includes only the information about the number of ratios and the value of the speeds and distances of these ratios. As a result, the input vector to trials of three tasks consists of a set of four ratios, whereas the input vector to trials of six tasks consists of a set of seven ratios. Each threat is actually a pair of speed and distance values. In case of three tasks, such an input vector noted x_3 will be described as follows:

$$x_3 = (d1/v1, d2/v2, d3/v3, d4/v4) \tag{6.1}$$

where d1, d2, d3, and d4 are the distances associated with ratios 1, 2, 3 and 4, and v1,v2, v3 and v4 are the speeds associated with the same ratios. An example of such an input vector may be the following:

$$x_{3,i} = (11/ 34, 25/89, 32/33, 28/57) \tag{6.2}$$

The values taken by the distances and the speeds are constrained by the requirements described in section 4.5. There are three types of sets: First, the set S_1 of possible speeds and distances, then R_1, the set of possible ratios where the speeds and distances belong to S_1. Finally X_3, and X_6, are the sets of possible combinations of ratios for three and six tasks. X_3 and X_6, may also be divided into subsets of ratios larger than one and subsets of ratios smaller than one, noted $X_{3|x<1}$, $X_{3|x>1}$, $X_{6|x<1}$, $X_{6|x>1}$, respectively.

The input alphabets are X_3 for trials of three tasks and X_6 for trials of six tasks. The ordering of the components of each input vector matters, i.e., the two vectors $x_{3,1}$ and $x_{3,2}$ are not considered identical.

$$x_{3,1} = (11/ 34, 25/89, 32/33, 28/57) \tag{6.3}$$

$$x_{3,2} = (25/89, 11/34, 32/33, 28/57) \tag{6.4}$$

The above vectors are different because the order in which the subjects process the ratios often has an impact on the final solution. The subjects use approximation methods to compare the ratios; as a result, when given the same ratios but in a different order, the probability of error is affected.

The input alphabets have been characterized. Now the distribution and the number of elements of the input alphabets X_3 and X_6 must be evaluated to compute the entropy of the input vectors x_3 and x_6.

The distribution of both alphabets is assumed to be uniform, because each input vector x_i is generated randomly. (It is assumed that each vector has the same probability of being generated.). The cardinal of each input alphabet is difficult to assess because of the constraints imposed on the ratios. Therefore these figures are estimated as follows. First the number of elements of each alphabet is computed assuming that there are no constraints on the sets of ratios. Then, a computer program is used to estimate the number of eligible combinations of ratios when the constraints are included.

The pool of acceptable ratios less than one is 3003, and the pool of acceptable ratios larger than one is 2407. (These figures were computed by generating every possible pair of distances and speeds and counting all the feasible ones. The number of ratios larger than one is less than the number of ratios less than one, because the ratios larger than one were subject to an additional constraint: they had to be larger than 1.2).

If the constraints imposed among combinations of ratios were ignored, the number of *input vectors less than one for three tasks* would be:

$$A_{3003}^{4} = 3003 * 3002 * 3001 * 3000 = 8.1162 * 10^{13} \tag{6.5}$$

and the number of input vectors larger than one would be:

$$A_{2407}^{4} = 2407 * 2406 * 2405 * 2404 = 3.3483 * 10^{13} \tag{6.6}$$

That is, ignoring the constraints imposed betweeen ratios, the size of the input alphabet X_4 would be:

$$A_{3003}^{4} + A_{2407}^{4} = 11.4645 * 10^{13} \tag{6.7}$$

The same way, ignoring the constraints imposed betweeen ratios, the size of the input alphabet X_6 would be:

$$A^{7}_{3003} + A^{7}_{2407} = 2.187 * 10^{24} + 4.634 * 10^{23} = 2.650 * 10^{24} \quad (6.8)$$

Such large input alphabets do not allow enumeration.

The program which was used to estimate the number of feasible input ratios was based on the method used to generate sets of ratios during the experiment. An iteration consisted of picking a distance and a speed satisfying the necesary constraints. Then the number of possible second ratios was computed by enumeration. A second ratio out of the pool of possible ratios was then picked randomly, and the number of possible third ratios was then computed... Following the same procedure for the remaining ratios, for each run, the program computed the number of possible second N_2, third N_3, fourth N_4..N_7 ratios. For each run i, for three tasks the number of possible combinations of ratios, noted P_{i3} could be derived as the following product:

$$P_{i3} = N_{i1} * N_{i2} * N_{i3} * N_{i4} \quad (6.9)$$

and for six tasks P_{i6}:

$$P_{i6} = N_{i1} * N_{i2} * N_{i3} * N_{i4}\ N_{i5} * N_{i6} * N_{i7} \quad (6.10)$$

The program was run 150 times for both ratios larger than one and ratios less than one. The estimated number of of possible first, second, third ..seventh ratios were derived for ratios larger than one and for ratios less than one for both number of ratios as follows:

Ratios <1

$$\bar{N}_{j\,|<1} = \left(\sum_{i=1}^{150} N_{ij\,|<1} \right) / 150 \quad j = 1 \text{ to } 7 \quad (6.11)$$

Ratios >1

$$\bar{N}_{j\,|>1} = \left(\sum_{i=1}^{150} N_{ij\,|>1} \right) / 150 \quad j = 1 \text{ to } 7 \quad (6.12)$$

Therefore, the size of the input alphabet X_3 could be derived as following:

$$C_{x3} = \prod_{i=1}^{4} \bar{N}_{i4 \,|\, <1} + \prod_{i=1}^{4} \bar{N}_{i4 \,|\, >1} \qquad (6.13)$$

The results for three tasks were the following:

$$C_{x3} = (3003*2567*2163*1793) + (2407*2355*2315*2276) \qquad (6.14)$$

$$C_{x3} = 2.9896*10^{13} + 2.9867*10^{13} = 5.9763 * 10^{13} \qquad (6.15)$$

The size of the input alphabet X_6, noted C_{x6} was derived using the same method as for X_3. The results were as follows:

$$C_{x6} = (3003*2567*2163*1793*1459*1161*913)$$

$$+ (2407*2355*2315*2276*2238*2202*2168) \qquad (6.16)$$

$$C_{x6} = 4.6236*10^{22} + 3.1910*10^{23} = 3.6534 *10^{23} \qquad (6.17)$$

The constraints imposed on the set of ratios also created difficulties when considering the internal variables which are described in section 6.1.2.

6.1.2 The Internal Variables

Before considering the entropy of the internal variables and the workload associated with each algorithm, the internal variables must be characterized. Therefore, as a first step, the subjects' approach to the experimental task and the level of detail used for modeling the algorithms are defined. Then the methodology used to assess the probability distributions of the internal variables is described.

Two different approaches were possible when modeling the experiment The subjects' tasks could be interpreted either as : 'to find the smallest ratio of a population sample' or as 'given four ratios, find the smallest'. In the first case, the distribution of the value of the smallest ratio when observing samples of four would have been the critical issue. In the latter case, the values of the smallest ratio would have been of no importance. Instead, the smallest ratio's position in the sequence (that is what is the first, second, third or fourth) would have been the required solution. The first approach was modeled in this thesis. The stategies that the subjects reported using were influenced by the values the ratios could take. Therefore models based on population samples seemed more appropriate. Another modeling issue related to short term and long term memory. With regard to short term memory, it is assumed that the decisionmakers are memoryless: they do not remember the approximated value of the ratio which was smaller in the previous comparison and must approximate it again for the following comparison. Such an assumption was derived after talking to subjects. They reported that they generally reestimated the ratios for each comparison. With regard to long term memory, it was assumed that the subjects could rank order the single digits ratios and did not need any special algorithm to do so.

The modeling approach has been discussed and the level of detail used in the models is now described. Within each algorihm, the different processes are kept as steps, but each operation required to perform the process is not recorded as a variable. This methodology keeps the number of internal variables under control; only the basic variables are recorded as variables. The internal variables of the first decision of Algorithm 1 for three tasks are described below in Figure 6.1, as an example.

The notation used in Figure 6.1 may be described as follows:

dij = jth digit of distance of ratio i. dij ranges from 1 to 9

$$w21 = \min(Ti,Tj) = \begin{cases} 0 \text{ if the two values are the same} \\ 1 \text{ if the first is smallest, Ti in this case} \\ 2 \text{ if the second is smallest, Tj in this case} \end{cases}$$

w22 = distance associated with w21, where w22 takes the value of the distance associated with the ratio corresponding to the value of w21. If w21 had taken a value of 1, w22 would take the values of d(Ri), since Ri would be smaller than Rj; such a ratio could be noted Ri'. If w21 takes a value of 0, each ratio (either Ri or Rj) has a probability of 0.5 of being chosen.

Input vector, X $X=(d1/v1, d2/v2, d3/v3, d4/v4)$

Internal Variables, wi

w1 = d1	w5 = v1	w9 = trunc(d1/10) = d11	w13 = trunc(v1/10) = v11
w2 = d2	w6 = v2	w10 = trunc(d2/10) = d21	w14 = trunc(v2/10) = v21
w3 = d3	w7 = v3	w11 = trunc(d3/10) = d31	w15 = trunc(v3/10) = v31
w4 = d4	w8 = v4	w12 = trunc(d4/10) = d41	w16 = trunc(v4/10) = v41

IF w17 d1<20 and v1>90 THEN Y = R1 END OF ALGORITHM
ELSE IF w18 d2<20 and v2>90 THEN Y = R2 END OF ALGORITHM

ELSE w19 = d11/v11 = T1
w20 = d21/v21 = T2

w21 = min(T1,T2)
w22 = distance of w21 = d(w21)
w23 = speed of w21 = v(w21)

NEXT COMPARISON

Figure 6.1 The Information Theoretic Description of Algorithm 1: The First Decision

The modeling process and the choice of internal variables have been described. The next step is to derive the probability distribution of each variable and compute the workload of each algorithm. First, however, the impact of two of the experimental setups on the probability distributions are discussed. The effect of having trials consisting of ratios either larger than one or less than one is described in section 6.1.3. Then, the assumptions required to evaluate the probability distributions are described in sections 6.2 and 6.3.

6.1.3 The Trials: Ratios Less than One and Ratios Larger than One

The trials were set up so that whether the ratios would be larger than one or less than one would be picked randomly. Such a setup had an impact on the distribution of the internal variables. There was a 0.5 probability that a trial would consist of ratios less than one, and a 0.5 probability that the trial would consist of ratios larger than one. Therefore, the entropy of an internal variable wi may be expressed as follows:

$$H(wi) = -\sum_{wi} p_{wi}(wi) \log_2 p_{wi}(wi) \tag{6.18}$$

where

$$p_{wi}(wi) = p_{wi|x<1}(wi|x<1)*p(x<1) + p_{wi|x>1}(wi|x>1)*p(x>1) \tag{6.19}$$

x is the ratio from which w_i is derived

$$p(x<1) = p(x>1) = 0.5 \tag{6.20}$$

If a variable wi can *only* be derived *either* from a ratio larger than one, or from a ratio less than one then exactly one of the two equations below holds (6.21 or 6.22).

$$p_{wi|x<1}(wi|x<1) = 0 \tag{6.21}$$

or

$$p_{wi|x>1}(wi|x>1) = 0 \tag{6.22}$$

The input vector, X, as well as the individual ratios (di / vi) are such variables. For such variables equation 6.18 may be rewritten as:

$$H(wi) = -\sum_{wi|x<1} p_{wi}(wi) \log_2 p_{wi}(wi) - \sum_{wi|x>1} p_{wi}(wi) \log_2 p_{wi}(wi) \tag{6.23}$$

Finally equation 6.23 for the input entropy or the entropy of the ratios may be simplified as follows:

$$H(wi) = -\sum_{wi|x<1} p_{wi|x<1}(wi|x<1) *p(x<1) \log_2 [p_{wi|x<1} (wi|x<1) *p(x<1)]$$

$$-\sum_{wi|x>1} p_{wi|x>1}(wi|x>1) *p(x>1) \log_2 [p_{wi|x>1} (wi|x>1) *p(x>1)] \quad (6.23)$$

As a result, the input entropy for three tasks becomes:

$$H(x) = 0.5 * \log_2 (2.9896*10^{13}) + 0.5* \log_2 (2.9867*10^{13}) +1 \quad (6.24)$$

$$H(x) = 22.3825 + 22.3818+1 = 45.764 \text{ bits} \quad (6.25)$$

Because of the experimental setup, for each variable, the distribution must be derived seperately for the input vectors of elements larger than one and those of elements less than one: two different probability distributions are obtained. Then, the two are combined as in equation 6.19 to evaluate the entropy of each variable of the algorithms..

6.2 THE COMPUTATION OF ENTROPY

6.2.1 The Approach

The internal variables have been described and some of the computational issues were raised in the previous section. This section describes the methodology followed to assess the entropy of each variable.

A normal procedure to compute the probability distribution of each internal variable is to use a computer program simulating a binning process to assess the histogram of each internal variable as all the possible inputs are fed to the program. The probability distribution is then derived from the histogram. For this particular case however, a binning process using every element of the input alphabet may not be used because of the size of the input alphabet. Therefore assumptions must be made to estimate the probability distribution of each variable. First the two "categories" of internal variables are described.

Then, the methodology to estimate the probability distribution is reviewed for each.

6.2.2 The Different Types of Variables

Two different types of variables may be identified within each algorithm: The variables for which the entropy may be computed without comparing two ratios, and the variables for which the entropy could only be computed after one or more of the comparisons were made. For simplicity, the first group will be called the static variables and the second the non-static variables. (In Figure 6.1, variables w1 to w18 are considered as static, whereas variables w19 to 23 are non-static.)

The static variables are variables that are repeated, and are the same for each four (or seven) ratios. The distribution of the static variables were computed for one ratio, taking all the possible ratios larger and less than one. Then the same distribution was assumed for each ratio. These variables reflect the size of the input, and as a result dominate when considering the entropy of the total system. The very large entropy of these variables tends to overshadow the decision variables of the algorithms.

The non-static variables describe three categories of variables: the decision process, the approximated value of the ratios which were chosen to be the smallest after a comparison, and the intermediate variables used to arrive at the approximated value. The probability distribution of each category of non-static variables was estimated using computer programs. The distribution of the non-static variables changes after each comparison.

6.2.3 The Entropy of the Static Variables: Assumptions and Methodology

In this section, the most important assumptions used to compute the entropy of the static variables are given, while the methodology used to compute the entropy of a few static variables is described.

The first static variables to be considered are the ratios before they are compared. The distribution among ratios less than one is assumed to be uniform. The same is valid for the ratios larger than one. This assumptions is used even though the constraints imposed on the ratios will make some ratios appear in sets more often than others. Let R be the pool of all feasible ratios, R_0 the pool of all feasible ratios less than one and R_1 be the pool of all feasible ratios larger than one. Then the above assumptions may be described as follows:

$$\forall\ r \in R,\ \ p(r \in R_0) = 0.5 = p(r \in R_1) \tag{6.28}$$

$$\forall\ r_a \in R_i,\ \forall\ r_b \in R_i,\ p_r(r_a) = p_r(r_b) \ \text{ for } i = 0, 1 \tag{6.29}$$

Also, the entropy associated with each ratio of a set x = (R1, R2, R3, R4) is assumed to be the same. It is assumed that the entropy of the ratios is independent from the order the ratios appear on the screen. The entropy for each ratio may be computed as follows:

$$H_R = -\sum_{R} p_R(R) \log_2 [\, p_R(R)] \tag{6.30}$$

where $R \in R$

$$H_R = 0.5 \log_2 (3003) + 0.5 \log_2 (2407) + 1 = 12.39 \text{ bits} \tag{6.31}$$

The distances and the speeds forming each ratio are the next static variables studied. It is assumed that the distances are independent from one another, but are not independent of the speed associated with them to form a ratio. The probability distribution among the different possible distance values is not uniform. The entropy of the distances and the speeds may be computed as follows:

$$H_{wi} = -\sum_{wi} p_{wi}(wi) \log_2 p_{wi}(wi) \tag{6.32}$$

where pwi(wi) was computed by iteration using the binning process, considering first all the possible ratios larger than one, then all the possible ratios less than one. Each time the value wi appeared, the frequency of wi was increased by one. The entropy was the following:

$$H_{wi} = 6.41 \text{ bits} \tag{6.33}$$

where wi is a speed or a distance associated with a ratio before this ratio has been compared to another ratio.

The same procedure was done to estimate the probability distributions of the first digit of both speeds and distances.

$$H_{wi} = 3.16 \text{ bits} \tag{6.34}$$

where wi is the first digit of a speed or a distance associated with a ratio before the ratio was compared.

It is assumed that all the internal variables derived from the speeds and distances were independent of the sequence of the ratios. (The first digits are an example of such derived internal vaiables.) Therefore, these variables are assumed to be equally distributed for all four ratios when considering trials of four ratios, and all seven ratios when considering trials of seven. For example, when considering Algorithm 1, which is shown in Figure 6.1, the sets of variables shown in Table 6.1 are equally distributed.

Table 6.1 Sets of Equally Distributed Variables

Variables	Corresponding Internal Variables
d1, d2, d3, d4	w1 to w4
v1, v2, v3, v4	w5 to w8
d11, d21, d31, d41	w9 to w12
v11, v21, v31, v41	w13 to w16
decide if di < 20 and vi > 90	w17, w18, w24, w32
di1/vi1, i =1 to 4	w19, w20, w28, w36

The probability distribution of the other static variables were derived using the binning process and the assumptions just described.

6.2.4 The Entropy of the Non-Static or Decision Variables: Methodology

The distribution of the non-static variables was computed differently for each algorithm, since these variables were algorit' m-specific. However, the same terminology may be used to describe the steps that were f lowed.

Within each algorithm, the first two ratios noted $R_1$and R_2 were approximated into T_1 and T_2 which are the variables compared for the first decision, D1. It is assumed that T_1 and T_2 are equally distributed. The distribution of the decision D1, as well as that of the minimum of T_1 and T_2 was found by first assessing the distributions of T_1 and T_2, then, finding the probability that T_1 would be smaller and finally by finding the probability distribution of the minimum of T_1 and T_2. The same procedure was continued until the fourth or seventh approximated ratio was compared to the minimum of the previous comparison. While such a procedure was followed to find the distribution of the decision variables, the same method was used to assess the distribution of the 'non-static' variables.

The probability that the approximated ratio x1with distribution p_{x1} be less than the approxiamted ratio x2 *with distribution* p_{x2} *was computed as follows:*

$$p(x1 < x2) = \sum_{\text{all } x1} p_{x1}(x1) \sum_{x1}^{\infty} p_{x2}(x) \tag{6.37}$$

The distribution of the min of two variables x1, x2, was computed as follows:

$$y = \min(x1, x2) \tag{6.38}$$

$$P_y(y) = p_{x1}(y) \sum_{y}^{\infty} p_{x2}(x2) + p_{x2}(y) \sum_{y}^{\infty} p_{x1}(x1) \tag{6.39}$$

These formulas were used to compute the entropy of the non-static variables of the different algorithms. The entropy of each variable is shown in Appendix F.

6.3 THE WORKLOAD FOR EACH ALGORITHM

This section first summarizes the most important assumptions regarding the assessment of the variables' probability distribution. Secondly, the numerical values of the workload are presented and discussed. Thirdly, the feasibility of the results is reviewed by checking the consistency between the algorithms. Finally, the assumption derived in Chapter III regarding the correspondence between the workload for three and for six tasks is discussed. The evaluation of workload allows the testing of the hypotheses concerning the bounded rationality constraint in Chapter VII.

6.3.1 The Most Important Assumptions

Many assumptions and approximations have been described in section 6.2. Each has been used in the computation of the total entropy of the appropriate algorithm(s) to evaluate the workload associated with each algorithm. The most important and the most critical were the following:

(1) Assume uniform distribution of the input.

(2) Assume uniform distribution of the ratios, i.e., each ratio has the same probability of occurring in an input.

(3) The distribution of the approximated ratios and all the intermediate steps to obtain the approximated ratios is based on the first two assumptions.

(4) After a given comparison, the rate of change in entropy of the similar types of non-static variables is assumed to be the same. The rate of change is defined as the ratio of the entropy of the non-static variable used for comparison i to the entropy of the same variable when used for comparison i-1. (Examples of similar types of non-static variables would be the first digits and second digits of the speed values, or the the actual distance values and the approximation of the distance values used to make the comparison.)

6.3.2 The Numerical Values

The workload for each number of ratios and each algorithm was computed following the methodologies described in section 6.2. The numerical values are summarized in Table 6.2. As one may see from the table, the value of the workload varies significantly from algorithm to algorithm. For three tasks the workload ranges from 165.62 bits to 275.58 bits and the mean is 235.03. For six tasks, it ranges from 297.92 to 513. 59 bits and the mean is 433.04 bits.

Table 6.2 The Workload Associated with the Algorithms

Algorithm	Workload Three Tasks (in bits)	Workload Six Tasks (in bits)
1	210.103	386.700
2	262.031	480.059
3	275.582	513.594
4	227.858	417.450
5	165.615	297.915
6	268.995	502.530

The variation among algorithms is weighted by the number of subjects who were associated with the algorithm. In Chapter V, each subject was assigned an algorithm which attempted to model the basic operations or approximations performed by the subject. Therefore, the average (over the subjects) workload required by the experiment may be computed by multiplying the number of subjects who "used" a given algorithm by the workload of this algorithm. The results, when considering the number of subjects associated with each algorithm, are summarized in Table 6.3.

Table 6.3. The Average Workload for the Experiment Over Subjects

	Three Tasks	Six Tasks
Average workload	243.625	450.270
Standard Deviation	40.353	79.057

6.3.3 Consistency Among the Algorithms

When looking at the workload for both three tasks and six tasks, the workload associated with Agorithm 5 is signficantly lower than that of the other algorithms (165.615 bits for three tasks and 297.915 bits for six tasks). Such a low workload is explained by the structure of the algorithm itself. The algorithm consists of comparing the difference between the speeds and distances of the two ratios. Such a process requires only two steps before making the comparison i.e., compute each difference, which drastically reduces the workload. The workload is not based on the number of steps, but on the entropy associated with each variable. Because many of the intermediate internal variables have very significant entropies, the number of intermediate steps required to transform the input into variables that may be compared plays a significant role in the total entropy. Such an observation is particularily true for Algorithm 5, which is very simple. It is also applicable to Algorithm 1 which requires a limited number of steps before the comparisons are made.

Algorithm 1 has a larger workload than Algorithm 5 (210.103 bits for three tasks, and 386.700 bits for six tasks versus 165.615 and 297.915 bits) but it is still lower than that of the other three algorithms. Six steps are required to transform two input ratios into two variables that may be compared: truncate each speed and each distance (4 steps), and then form each single digit ratio (two extra steps). The other algorithms require a significant number of steps before a comparison is made.

The fact that algorithms 1 and 5 have smaller workload than the other three is explained by their structure. Another method to check the results of the workload values is by looking at the 3 different categories of algorithms which were derived in Chapter V.

The first category included algorithms 1 and 2 in which the ratios were transformed into single digit ratios and were compared. Algorithm 2 was defined as requiring more processing than Algorithm 1 since for the first case the rounded ratios are compared whereas in the other case the truncated ratios are compared. The computations of workload confirmed the expectations, the workload for Algorithm 2 is larger than that for Algorithm 1 (210.103 bits versus 262.031 bits for three tasks and 386.700 bits versus 480.059 bits for six tasks, an increase of 24.7 % for three tasks and 24.1 % for six tasks).

The second category of algorithms included algorithms 4, 5 and 6. The workload for Algorithm 4 is larger than that for Algorithm 5. The same structure is used, but Algorithm 4 computes four differences as opposed to two and makes two comparisons as opposed to one. The increase of workload was very significant, 37.6% for three tasks, and 40.1% for six tasks. Such an increase could be expected since the amount of internal processing is almost doubled. Algorithm 6 is a combination of algorithms 2 and 5. It uses the first steps of Algorithm 5 to determine if a small ratio could be spotted before any computation. If the test is not relevant, it rounds each ratio using the same methodolgy as Algorithm 2. The workload for Algorithm 6 was slightly larger than that for Algorithm 2 as expected, (268.995 bits versus 262.031 bits for three tasks, and 502.530 bits versus 480.059 for six tasks.) The increase of 2.8% for three and 4.6% for six tasks is small. The testing variables used in Algorithm 6 (and not present in Algorithm 2) have entropies of a few bits only.

Finally, Algorithm 3 is a seperate category since a different strategy is used for ratios less than one and larger than one. As a result, the number of internal variables is significantly increased even though each comparison requires only six intermediate variables (as Algorithm 1), two of which have entropies less than 2. Because of the different strategies for ratios less and larger than one, the workload for Agorithm 3 is the largest of all.

From the above remarks, it appears that the values for the workload are consistent between the algorithms. As a result the relative differences between the workload of the different algorithms are feasible and conclusions relating the different algorithms and their 'users' may be derived based on these values. The next step is to compare the workload for the same strategies, but for the different number of tasks within a trial.

6.3.4 Comparing the Workload for Three and Six Tasks

In Chapter III, it was postulated that the important parameters were not the number of ratios but the number of tasks. The assumption was: the workload per comparison is approximately the same for three and six tasks i.e., the workload for six tasks should be twice that for three tasks. The experimental results seemed to confirm this assumption since the T* values for three and six tasks were not significantly different. This section first shows the ratio of workload for three and six tasks for each algorithm. Then the values obtained are discussed and explained, and the validity of the assumption is assessed. Finally, a simple linear regression modeling the workload as a function of the number of tasks is presented .

The analytical results confirm the assumption that the workload for six tasks is approximately twice that for three tasks. On average, the ratio of the workload for six tasks to that of three tasks is close to 1.84. Table 6.4 shows the ratio for the six algorithms as well as the average over the six algorithms and the average when introducing the frequency of each algorithm.

Table 6.4 The ratio of the Workload for Six Tasks to that of Three Tasks

Algorithm #	Ratio (Six Tasks / Three Tasks)	Average Over Subjects
1	1.841	1.845
2	1.832	
3	1.864	
4	1.799	
5	1.868	
6	1.887	
Average Over Algorithms	1.839	

The fact the the workload for six tasks is not twice that for three tasks should not be regarded as unwanted noise. Such a 'discrepancy' is derived from the analytical models. First the entropy of the input is not proportional to the number of comparisons and does not increase linearly with the number of ratios because of the log function. The input for three tasks is 45.76 bits and for six 77.68 bits). Then, the internal variables increase this difference even more because the entropy of more than half of the internal variables reflect the entropy of the very large input alphabet. Finally, when considering the distribution of the minimum of two equally uniformely distributed variables (these were the assumptions used), it will be skewed towards the smallest values. This is particularly relevant to our experiment when considering the distribution of the min as the number of comparisons increases. The previous paragraph may be described analytically as follows:

Let X be an ordered population uniformly distributed and let N be the size of the population. Then

$$p_x(x) = \begin{cases} \frac{1}{N} & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \tag{6.40}$$

Let $y = \min(x_1, x_2)$ where x_1, x_2 are two elements of X, and f_y the distribution of y then :

$$f_y(y) = \begin{cases} \frac{2}{N}(1 - \frac{y}{N}) & \text{if } y \in X \\ 0 & \text{otherwise} \end{cases} \tag{6.41}$$

Let $z = \min(y, x_3)$, $x_3 \in X$ and g_z the distribution of z, then

$$g_z(z) = \begin{cases} \frac{3}{N}(1 - \frac{z}{N})^2 & \text{if } z \in X \\ 0 & \text{otherwise} \end{cases} \tag{6.42}$$

The distribution of the variable $t \in X$ being the smallest of the n^{th} comparison and a variable $u \in X$ is:

$$f_t(t) = \begin{cases} \frac{n}{N}(1 - \frac{t}{N})^{n-1} & \text{if } t \in X \\ 0 & \text{otherwise} \end{cases} \tag{6.43}$$

As an analogy to our experiment, x_1 and x_2 may be assumed to be the first two ratios to be compared. y takes the values of the ratios kept from the first comparison, x_3 is the third ratio to be compared, z, takes the values of the ratios kept from the second comparison ect... The distributions become more and more skewed, thereby reducing the entropy of the minimum after each comparison.

The decrease in entropy after each comparison ranges between 2% and 5% of the non-static variables. This is not very significant when considering the entropy of the whole system and the entropy of the static variables which are not affected by the decrease due to the comparisons. Also, in this particular case, the entropy related to the large input tends to dominate the entropy of the system and absorb the changes due to the decrease of the entropy of the decision variables (called non-static variables).

A simple least squares fit using the twelve data points of Table 6.2 (three and six tasks, algorithms 1 through 6),

$$Y_i = a X_i + b \tag{6.44}$$

where

$$X_i = 3, ..., 3, 6, ..., 6$$

$$Y_i = 210.03, 262.031..., 268.995, \ 386.700, 480.059, ...502.530$$

yields

$$Y = 66 X + 37 \tag{6.45}$$

For $X = 3$ $\quad Y = 235$
$X = 6$ $\quad Y = 433$

Note that 37 is equivalent to about half the effort of a comparison and is not very significant either for three or six comparisons. Because of the very few data points used (twelve), this regression should only be considered as a gross model, but it is important to note that the results are consistent with the other observations.

Therefore, considering all the assumptions which have been made throughout this thesis, the analytical results do not contradict the experimental results. The assumption made in Chapter IV was reasonable: the workload per comparison is approximately the same for three and six tasks.

The workload was evaluated for each algorithm and the values were consistent both between algorithms and with the experimental results. Therefore, these values may be used to assess the bounded rationality constraint for each subject and test hypotheses about the stability of F_{max} both across subjects and across tasks.

CHAPTER VII

THE BOUNDED RATIONALITY CONSTRAINT: RESULTS AND ANALYSIS

This chapter derives the bounded rationality constraint for each subject and studies its behavior. First, the hypotheses regarding the stability of F_{max} are stated. Then, the methodologies used to evaluate F_{max} and to test the hypotheses are described. Next, F_{max} is evaluated for each subject and each type of trials, three and six tasks. Finally the validity of the hypotheses are tested and the results are compared to the postulations made in Chaper IV.

7.1 THE HYPOTHESES

Two hypotheses concerningthe stability of F_{max} are to be confirmed.

Hypothesis (1). F_{max} is stable for an individual when minor tasks changes are made.

Hypothesis (2). F_{max} is stable across individuals and across tasks.

7.2 METHODOLOGIES

7.2.1 The Procedures to Evaluate F_{max}

In Chapter IV, the minimum average time required to perform the experiment was derived for each subject using the experimental results. In Chapter VI, the workload associated to each model was evaluated. The bounded rationality constraint which is noted F_{max} may now be computed for each subject and for both types of trials combining the experimental and the analytical results.

As described in section 2.3 , F_{max} is the ratio of the workload associated to the trial to the time threshold T*. Since the values of T* were evaluated as a time per task, the value of T* has to be multiplied either by three or six to consider the total duration of the trials. Therefore, for each subject and for both number of tasks, the value for the bounded rationality constraint may be computed as follows:

$$F_{max,i,j} = G_{i,j} / [j*T^*_{i,j}] \quad (7.1)$$

where

i is the subject number and j is the number of tasks

$G_{i,j}$ is the workload of the algorithm associated to subject i for j tasks

$T^*_{i,j}$ is the threshold processing time associated to subject i for j tasks

7.2.2 The Procedures for Testing the Hypotheses

The methodologies used to test the hypotheses are very similar to the methodologies used to test for the stability of T* across trials and across subjects.

To test the stabity of F_{max} *across trials,* first the distributions of $F_{max,3}$ and $F_{max,6}$ are assessed using a statistical test (the Chi-Square test) and are then compared. If the two distributions are of the same type, then it is tested if the mean of the two distributions are significantly different using a statistical test, (the t test).

The second hypothesis: the stability of F_{max} across trials and subjects is more simple to confirm. First, an F_{max} value is estimated for each subject, (for each subject, F_{max} is the average of $F_{max,3}$ and $F_{max,6}$). Then, a Chi-Square test is used to estimate whether the F_{max} distribution is significantly different from the normal distribution or not. A non-significant difference would lead to the conclusion that F_{max} is stable both across subjects and tasks.

7.3 COMPUTATION OF F_{max}

The values of F_{max} were computed for each subject for both number of tasks and are shown in Table 7.1 and were summarized in Table 7.2. The average value of $F_{max,j}$ over subjects is 44.35 bits/sec for three trials versus 41.00 bits/sec. for six trials. The standard deviation for three tasks is quite large 15, as is the one for six tasks, 13. It is interesting to notice that in both cases the standard deviation is almost one third of the mean.

Table 7.1 The F_{max} Values for Each Subject and Both Numbers of Tasks

Subject #	$F_{max,3}$	$F_{max,6}$
20	42.776	30.714
21	47.036	32.636
22	83.378	64.422
23	64.838	46.516
25	25.896	23.631
26	38.380	26.350
27	45.704	43.714
28	49.510	41.220
29	28.214	22.549
31	42.719	26.839
33	31.605	29.064
34	36.016	61.100
35	27.241	35.911
36	38.124	34.798
37	30.595	31.217
38	17.310	24.954
39	44.786	44.392
41	54.397	62.652
44	65.718	55.087
45	42.096	29.775
46	28.737	23.903
50	45.150	44.840
51	31.113	42.148
52	64.684	54.414
53	40.672	42.081

Table 7.2 Summary of the F_{max} Values for Both Numbers of Tasks

	$F_{max,3}$ (bits/sec)	$F_{max,6}$ (bits/sec)
Average	42.668	38.997
St. Dev.	15.068	12.873
Min	17.310	22.549
Max	83.378	64.422

It is important to realize however, that the values obtained for the bounded rationality constraint are not of any specific interest if just considered as values. The different algorithms that could be used to model the same task could increase the workload, and therefore F_{max} as well by a factor of two or more. Therefore, it is by studying the distribution of F_{max} as the tasks is slightly changed, and across subjects, as well as by comparing the conclusions derived analytically with the conclusions derived experimentally that the significant conclusions may be derived. As long as each algorithm is modeled consistently with the others, the comparisons may be done.

7.4 TESTING THE HYPOTHESES

7.4.1 The stability of F_{max} Across Trials

To test the stability of F_{max} across trials, the distribution of $F_{max,3}$ and $F_{max,6}$ must first be evaluated. In Chapter IV, it was established that the T* values were normally distributed for both three and six tasks and it had been postulated that the distribution of the T*'s should be closely related to that of F_{max}. This postulation was confirmed: goodness of fit tests showed that the distribution of both $F_{max,3}$ and $F_{max,6}$ were normal. (The Q^2 error was 2.0 for three trials and only 0.8 for six trials. See details in Appendix D). Figure 7.1 shows the distribution of $F_{max,3}$ over subjects, and Figure 7.2 shows the frequency distribution of $F_{max,6}$. The difference between the normal distribution and that of the $F_{max,3}$ values is shown in Figure 7.1, whereas the difference between the normal distribution and the $F_{max,6}$ values is shown in Figure 7.2. (Notice that the size of the intervals are not the same. The intervals are constructed as for the Chi-Square test: the cumulative probability within each interval is 0.2, see Appendix D.)

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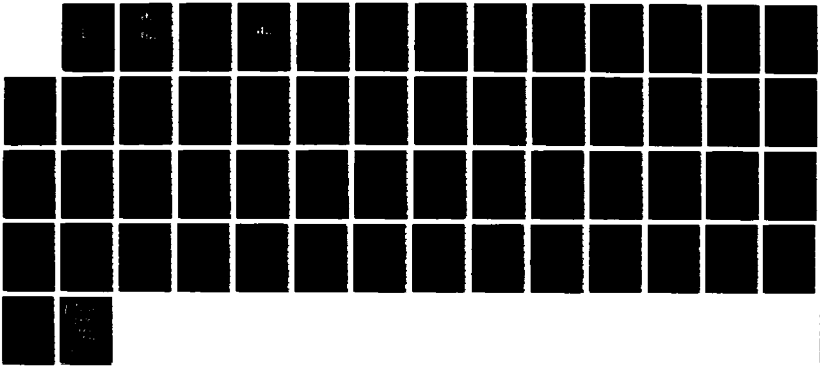

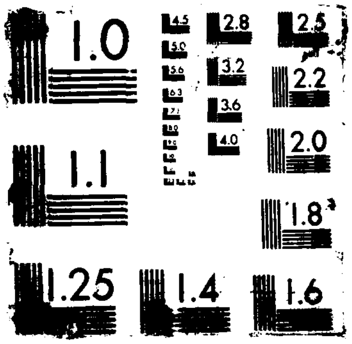
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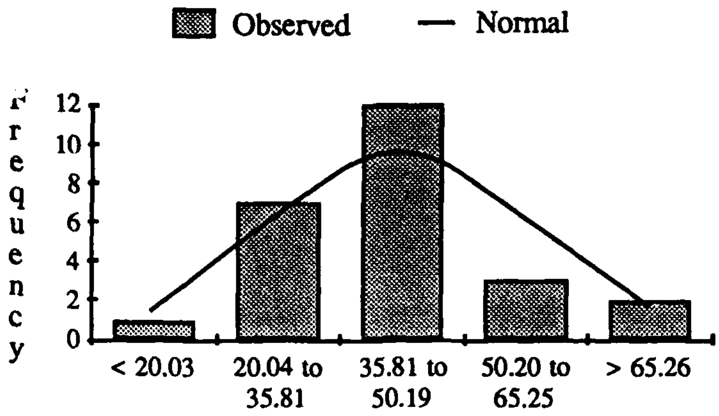

Figure 7.1 The Distribution of F_{max} for Three Trials

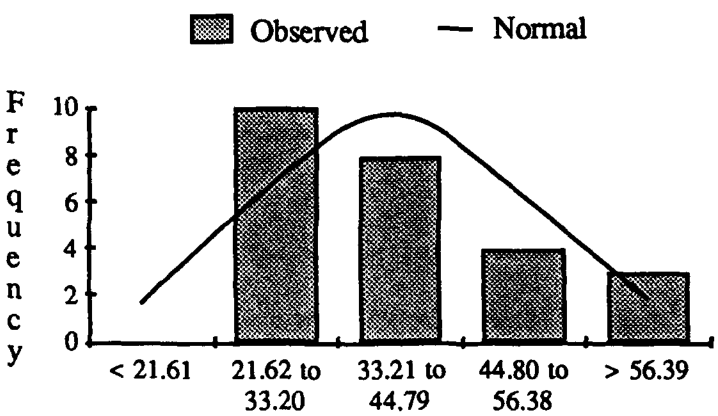

Figure 7.2 The Distribution of F_{max} for Six Trials

The next step needed to validate the hypothesis that F_{max} is stable across tasks is to compare the means of the $F_{max,3}$ and $F_{max,6}$ distributions. The experimental results had postulated that F_{max} was not significantly different for trials of three and six tasks. This result was confirmed by a statistical t test. The value for the statistical t test was 1.79 . The critical value for a two sided t test at a 0.95 level of confidence with 24 degrees of freedom is 2.06; 2.06 is larger than 1.79, so the hypothesis that the two distributions are of same mean may not be refuted . (See additional details in Appendix D.)

Therefore, one may say that F_{max} is stable for each subject as the number of tasks is varied from three to six. As a result, it may be assumed that there is only one significant value for each subject, which will be taken as the average of the F_{max} 's for three and six tasks.

In addition, these results provide indirect evidence for the stability of F_{max} over time, since each subject was tested on three or four different days. (A "composite" curve resulting from wide day to day fluctuations in the bounded rationality constraint would not likely reveal a clear threshold.) This stability suggests that it may not be necessary to measure a decision maker's F_{max} value for every type of task the decision maker may have to perform. Instead, the decision maker's F_{max} value could be measured using a prototypic "calibration" task. The value obtained from this prototypic task could be safely assumed to apply to a substantial range of structurally similar tasks.

7.4.2 The Stability of F_{max} Across Subjects

The next step of this Chapter is to study the behavior of F_{max} over all subjects. The F_{max} associated with each subject i was computed as follows:

$$F_{max,i} = \sum_{j=3,6} F_{max,i,j} \,/\, 2 \tag{7.2}$$

for i = 1 to 25

The F_{max} values were summarized in Table 7.3. A Goodness of fit test showed that the distribution was not significantly different from normal (the error, see Appendix D is $Q^2 = 5.2 < \chi_{0.95,2} = 5.99$). Therefore, it may be assumed that the distribution of F_{max} over subjects is stable, and the analytical results confirm the experimental results. Figure 8.3 shows the disribution of the individual values of F_{max}.

The analytical results have confirmed the experimental results. The bounded rationality not only exists for all the subjects, but it is uniformly distributed for each type of trials over the subjects, it is stable to minor tasks changes, and finally it is also uniformly distributed when assuming only one F_{max} value for each subject.

Table 7.3 Summary of the Average F_{max} Values over Subjects
(in bits per sec.)

Mean	40.830
Standard Deviation	13.013
Min	21.132
Max	73.906

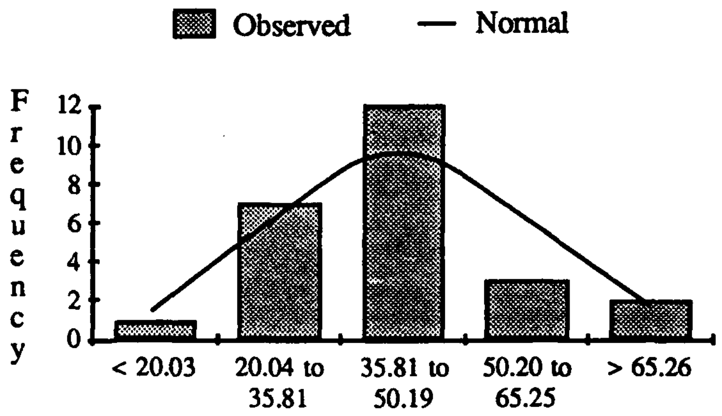

Figure 7.3 Distribution of the Average F_{max} Values over Subjects

When considering a particular task performed by well trained decisionmakers, it may be assumed that despite the individual differences and the different algorithms used, the bounded rationality is uniformly distributed among people. One could submit the hypothesis that in a very strict environment such as the military, where people who perform the same job should all be very qualified, the distribution of individual bounded rationality constraint for similar tasks would not only be normal but also extremely peaked. This could help significantly when designing organizations where the decisonmakers are not to be overloaded.

CHAPTER VIII

CONCLUSIONS AND FUTURE RESEARCH

8.1 CONCLUSIONS

8.1.1 The Thesis in Review

Both the analytical and experimental results were needed to answer most of the questions related to the bounded rationality constraint of human decision makers. The first significant results are derived from the experimental analysis in Chapter IV. First the existence of the bounded rationality constraints was proved. Second, the minimum time required to make one comparison on average, (noted T^*_3 for three tasks and T^*_6 for six) were identified for each subject. Finally, from the distribution of the T* values, postulations were made about the two hypotheses which were still to be tested: the stability of the bounded rationality constraint both across similar tasks and across subjects. The first step in confirming these postulations is made in Chapter V where algorithms representing models of the subjects' decision processes are identified and their plausibility is tested. Then, the workload associated with each algorithm is computed in Chapter VI. Finally, in Chapter VII, the experimental and analytical results are combined to derive the value of the maximum processing rate for each subject both for trials of three and six tasks. The hypotheses are then tested: the bounded rationality does not only exist but it is both stable across similar tasks and across subjects.

8.1.2 Applicability of Information Theory

Information theory was the mathematical tool used to assess the amount of cognitive workload required to perform the experiment given the different algorithms that were modeled. The workload associated with the different algorithms was consistent with the complexity of the algorithms and the different categories of algorithms. Such a result gave some validation of the mathematical model used. When trying to model the difference between the number of ratios, there was a slight discrepancy between the experimental and analytical results. Three postulations were made to explain the slight difference. First, the model for three and six tasks might not have captured the different approach that the subjects might have taken during the experiment. When assessing models in Chapter V, it was found that simulations of the models for six tasks consistently predicted worse performance than the subjects', whereas the performance was very similar when considering three tasks. Second, considering the very large size of the input alphabet, it is possible that the subjects did not recognize that the probability distribution of some of the variables were changing as the number of ratios to consider increased; the subjects might not have changed their strategy accordingly. Third, it should not be forgotten that the experimental results, particularly the T*'s were artificially constructed from the data, and therefore necessarily introduced some marginal errors in the experimental results. Finally, other factors such as time allocation, or short term memory may have affected the workload, but these factors are beyond the scope of this thesis. Because not a single subject mentioned using a different approach when processing trials of three and six comparisons, the models described in this thesis are reasonable considering the small discrepancy.

8.1.3 The Existence of the Bounded Rationality Constraint

The existence of a bounded rationality constraint for each subject was proved from the experimental results. Performance was fairly stable before it dropped rapidly. The S curves which were used to model the experimental results erased any discrepancy and at the same time any change in strategy which might have been apparent otherwise. Therefore, it may be postulated that the individual T*'s which were constructed graphically, represented an average over several t*'s , each associated with a given algorithm requiring a certain amount of cognitive workload. The individual t*'s were not identifiable, therefore, the value which was retained was the T*. The T* value was also considered as the critical value (instead of any possible t*), because the workload surrogate as computed using information theory, requires that the processors be above the bounded rationality constraint, and such was not possible to assert for the t*'s. The algorithm associated with each T* was supposed to be the algorithm corresponding to trials for which enough time was allowed for processing the task.

8.1.4 The Stability of F_{max} Across Tasks and Across Subjects

Both the experimental and analytical results confirmed the stability of F_{max} across similar tasks and across subjects. However, when comparing the experimental and analytical results, it appeared that the stability of $F_{max,3}$ and $F_{max,6}$ over subjects (both distributions are normal) was a more reliable result than the stability of the individual F_{max} across subjects. (The Q^2 value was larger for the F_{max} distribution than for the $F_{max,4}$ and $F_{max,7}$ distributions). This slight difference is derived from the discrepancy between the workload per comparison for trials of three and trials of six tasks. One may conclude however that F_{max} is stable across tasks for each individual, across individuals for each type of task, and finally that F_{max} is stable when considered simultaneously across tasks and across individuals. Considering the nature of the experiment, (the size of the input alphabet which did not allow enumeration), the number of different strategies that could be used to perform the task, the speed at which some of the subjects were capable to perform the task, the obtained results were very significant.

8.2 FUTURE RESEARCH

This experiment is only the first in a series of experiments trying to analyze and quantify the bounded rationality of human decisionmakers under pressure. The task which was analyzed was very basic and included only a single decisionmaker. Research has been undertaken at the Laboratory for Information and Decision Systems at MIT to design multi-person experiments and both validate some of the results obtained in this thesis on a multiperson level and derive other conclusions on the behaviour of the bounded rationality constraint. When considering multi person organizations, the impact of one DM being overloaded on the performance on the organization as a whole is also an interesting topic to investigate. In the latter case, the different organization structures should be studied.

When considering single person organizations, several issues which were raised in this thesis but not explored thoroughly could be investigated. The first topic relates to the small discrepancy which has appeared between the experimental results and the analytical results. In particular, the question concerning the different approaches to a seemingly similar task should be raised and explored further. How can the fact that subjects seem to consider making three or six comparisons as just twice the same task, (the T*'s for three and six tasks were similar) be modeled or predicted? Which other factors were involved and not considered by the models? The second topic relates to the nature of this experiment. The very large input alphabet only allowed approximations when computing the entropy, and the simple task permitted many different strategies which were not clearly identifiable. Running a single person experiment but with a more complex task which would allow fewer strategies and involve long term memory to a lesser extent could be considered. An other interesting task would be to analyze the experimental results using a different methodology to assess the T*'s and test if the conclusions still hold as strongly. Finally, the change in strategies as the time allotted to perform the tasks is decreased should be investigated. The slope at which the performance decreased should give a reasonable indication of the coping strategies (if any) that each subject used to behave toward the increasing time pressure. One could investigate the implication of subjects switching strategies as the time allotted per trial decreased on the evaluation of F_{max}.

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APPENDIX A

THE PARAMETERS OF THE GOMPERTZ FIT FOR EACH SUBJECT AND BOTH NUMBERS OF TASKS

Table A.1 Three Tasks: The Parameters of the Gompertz Fit for Each Subject

Subject #	a	b	c
20	0.6953	1.6076	1.0212
21	0.6010	9.5199	2.1388
22	0.5821	121.0751	7.1516
23	0.5425	23.2790	4.1538
25	0.6623	4.1534	1.4740
26	0.5540	3.9367	1.6926
27	0.8064	5.6502	1.7165
28	0.6656	3.0943	1.5349
29	0.6773	1.4991	1.0851
31	0.5414	222.7760	4.0071
33	0.6080	2.3370	0.9289
35	0.7903	2.0479	0.9781
36	0.6358	4.1795	1.5536
37	0.7101	2.3430	1.0926
38	0.6431	2.7444	1.5118
39	0.5910	4.0600	0.7710
40	0.7374	3.5761	1.4590
41	0.6100	2.9660	1.7020
44	0.6700	7.5195	2.6726
45	0.5908	17.1299	2.1973
46	0.6125	4.7761	1.0447
50	0.6683	6.3880	1.7990
51	0.8295	0.9083	0.5494
52	0.6771	20.3327	3.4126
53	0.6100	4.1469	1.4246

Table A.2 Six Tasks: The Parameters of the Gompertz Fit for Each Subject

Subject #	a	b	c
20	0.7733	2.1530	0.8917
21	0.6825	8.8417	1.5899
22	0.5939	82.9349	5.6819
23	0.5774	3.3351	1.9541
25	0.6242	5.7319	1.6488
26	0.6643	2.6720	1.1043
27	0.8147	8.0274	1.9412
28	0.6932	2.0653	1.1767
29	0.7224	2.1707	1.1323
31	0.5843	11.1012	1.5914
33	0.6706	4.7400	1.1894
35	0.6987	9.9133	2.9269
36	0.5592	31.5352	3.7387
37	0.6954	2.6711	1.1221
38	0.6740	3.2513	1.8216
39	0.4214	71.0487	2.3160
40	0.6993	9.0014	2.0307
41	0.7177	9.8930	2.9997
44	0.7139	7.6628	2.4163
45	0.7106	12.7590	1.5870
46	0.8040	6.4231	1.0212
50	0.6613	6.1570	1.8930
51	0.7381	2.2252	1.2399
52	0.6361	14.7120	2.8631
53	0.7177	6.2944	1.7876

APPENDIX B

THE R^2 VALUES : THE GOMPERTZ VERSUS THE LINEAR FIT

Table B1 The R^2 Values for each Subject for Both the Linear and the Gompertz Fit

Subject #	Three Tasks		Six Tasks	
	S Curve	Line Curve	S Curve	Line Curve
20	0.993	0.790	0.984	0.766
21	0.984	0.730	0.993	0.897
22	0.985	0.252	0.992	0.452
23	0.999	0.424	0.988	0.501
25	0.989	0.785	0.994	0.829
26	0.990	0.662	0.992	0.828
27	0.985	0.780	0.989	0.782
28	0.989	0.733	0.991	0.741
29	0.992	0.717	0.992	0.826
31	0.967	0.720	0.972	0.847
33	0.972	0.640	0.989	0.879
35	0.988	0.798	0.987	0.656
36	0.990	0.793	0.986	0.525
37	0.990	0.774	0.977	0.653
38	0.932	0.668	0.986	0.664
39	0.952	0.668	0.927	0.864
40	0.991	0.773	0.987	0.804
41	0.982	0.719	0.984	0.849
44	0.971	0.457	0.993	0.541
45	0.977	0.791	0.982	0.891
46	0.967	0.861	0.968	0.923
50	0.986	0.749	0.995	0.827
51	0.987	0.749	0.983	0.827
52	0.985	0.290	0.996	0.627
53	0.991	0.826	0.965	0.625

APPENDIX C

THE T* VALUES

Table C1. The T* Values for Both Numbers of Tasks for Each Subject

Subject #	T^*_4	T^*_7
20	2.147	2.787
21	1.857	2.452
22	0.911	1.080
23	1.171	1.496
25	2.132	2.101
26	1.825	2.446
27	2.010	1.958
28	1.855	2.077
29	1.957	2.202
31	1.778	2.592
33	2.764	2.753
35	2.490	1.371
36	2.027	1.383
37	2.352	2.407
38	1.804	1.591
39	4.046	2.583
40	2.051	1.928
41	1.648	1.337
44	1.398	1.554
45	2.075	2.687
46	3.141	3.504
50	1.986	1.868
51	2.952	2.031
52	1.386	1.539
53	2.205	1.990

APPENDIX D

THE DIFFERENT STATISTICAL TESTS: PROCEDURES AND RESULTS

D.1 THE GOODNESS OF FIT TESTS: THE χ^2 TESTS

D.1.1 Overview

Goodness of fit tests were used to test whether the distribution of the T*'s and the the distribution of the F_{max} 's for both fthree and six tasks were normally distributed. The test is done as follows:

$$Q^2 = \sum_{i=1}^{n} (\text{Expected}_i - \text{Observed}_i)^2 \; / \; \text{Expected}_i \qquad \text{(D.1)}$$

where Q^2 is the deviation error form the normal, n is the number of intervals chosen, Expected_i is the expected frequency in interval i if the distribuiton was normal, and Observed_i is the oberved frequency in interval i.

The intervals were constructed around the mean, using $\pm$ 0.842 and $\pm$ 0.253 as multipliers of the standard deviation to obtain five intervals with a probability density of 0.2. As a result the expected frequency per class for a normal distribution would be 5, and the assumptions necessary to perform a Chi-Square test would be satisfied. (A minimum expected frequency of 5 per class is required to perform a goodness of fit test.)

For the distribution to be accepted as normal, the Q^2 value must be less than the χ^2 value corresponding to the level of confidence chosen and the degrees of freedom. For 0.95 level of confidence and 2 degrees of freedom (2=5-2-1) we have :

$$\chi^2_{0.95,2} = 5.99 \qquad \text{(D.2)}$$

D.1.2 The Goodness of Fit Tests for the Different Distributions

For each distribution, a goodness of fit test was used to establish whether the distributions were different from normal. Tables D1 through D5, show the detailed analysis used for the different distributions.

Table D.1 The Chi-Square Test for the Distribution of T_3^*

Ranges for T_3^* (in sec.)	Observed Frequency	Expected Frequency	Error
< 1.53	4	5	0.2
1.54 to 1.91	6	5	0.2
1.92 to 2.24	9	5	3.2
2.25 to 2.63	2	5	1.8
2.64 >	4	5	0.2
Total	25	25	5.6

$Q^2 = 5.6 < 5.99$. Therefore, the distribution of T_3^* is not significantly different from the normal.

Table D.2 The Chi-Square Test for the Distribution of T_6^*

Ranges for T_6^* (in sec.)	Observed Frequency	Expected Frequency	Error
< 1.57	7	5	0.8
1.58 to 1.92	2	5	1.8
1.93 to 2.21	7	5	0.8
2.22 to 2.56	3	5	0.8
2.57 >	6	5	0.2
total	25	25	4.4

$Q^2 = 4.4 < 5.99$. Therefore, the distribution of T_6^* is not significantly different from the normal.

Table D.3 The Chi-Square Test for the Distribution of $F_{max,3}$

Ranges for $F_{max,3}$ (in bits)	Observed Frequency	Expected Frequency	Error
< 29.98	5	5	0.0
29.99 to 38.88	6	5	0.2
38.86 to 46.48	7	5	0.8
46.49 to 55.35	3	5	0.8
55.45 >	4	5	0.2
Total	25	25	2.0

Q^2 =2.0 < 5.99. Therefore, the distribution of $F_{max,3}$ is not significantly different from the normal.

Table D.4 The Chi-Square Test for the Distribution of $F_{max,6}$

Ranges for $F_{max,6}$ (in bits)	Observed Frequency	Expected Frequency	Error
< 28.16	6	5	0.2
28.17 to 35.74	6	5	0.2
35.75 to 42.25	4	5	0.2
42.26 to 49.84	4	5	0.2
49.85 >	5	5	0.0
Total	25	25	0.8

Q^2 = 1.2 < 5.99. Therefore, the distribution of $F_{max,6}$ is not significantly different from the normal.

Table D.5 The Chi-Square Test for the Distribution of the Average F_{max}

Ranges for F_{max} (in bits)	Observed Frequency	Expected Frequency	Error
< 29.87	4	5	0.2
29.88 to 37.54	9	5	3.2
37.55 to 44.12	2	5	1.8
44.13 to 51.79	5	5	0.0
51.80 >	5	5	0.0
Total	25	25	5.2

$Q^2 = 5.2 < 5.99$. Therefore, the F_{max} distribution is not significantly different from the normal .

D.2 THE t TEST: AN OVERVIEW

A t test was used to determine whether the mean values for T_3^* and T_6^* were significantly different. The same test was used for $F_{max,3}$ and $F_{max,6}$

Before a t test was run, it was established using the Chi-Square test that both distributions were of the same type: in each case the distributions were normal. Then, the t test for dependent distributions was used. The hypotheses Ho_1 and Ho_2, were as follows:

Ho_1: the means of the $T^*_{3,i}$ and $T^*_{6,i}$ are equal, i.e., the distribution $T^*_{3,i} - T^*_{6,i}$ has a mean not significantly different from 0.

Ho_2: the means of the $F_{max,3,i}$ and $F_{max,6,i}$ are equal, i.e. the distribution $F_{max,3,i} - F_{max,6,i}$ has a mean not significantly different from 0.

A two sided test was performed. For each test, the t value was computed as follows:

$$t = \text{sample mean } (T^*_{3,i} - T^*_{6,i}) \,/\, (\text{sample variance } / \text{sqrt}(n)) \tag{D.1}$$

The critical t value for a 95% level of confidence and 24 degrees of freedom is $t^*_{24,0.25} = 2.064$. If

$$-2.064 < t < 2.064 \tag{D.2}$$

then it was concluded that the two distributions were not significantly different.

The t value for the T* distributions was 0.1, whereas for the F_{max} distributions it was 1.69. Therefore both Ho_1 and Ho_2 are true.

APPENDIX E

THE SUBJECTS AND THE ALGORITHMS

Table E1. The Subjects' Performance Versus the Algorithms'

Subject #	Algorithm #	Three Tasks		Six Tasks	
		Subject	Algorithm	Subject	Algorithm
20	3	0.695	0.719	0.773	0.564
21	2	0.601	0.665	0.683	0.580
22	4	0.582	0.558	0.594	0.486
23	4	0.543	0.558	0.577	0.486
25	5	0.662	0.627	0.624	0.530
26	1	0.554	0.654	0.664	0.559
27	3	0.806	0.719	0.815	0.564
28	3	0.666	0.719	0.693	0.564
29	5	0.677	0.627	0.722	0.530
31	4	0.541	0.558	0.584	0.486
33	2	0.608	0.665	0.671	0.580
35	6	0.790	0.692	0.699	0.635
36	5	0.636	0.627	0.559	0.530
37	6	0.710	0.692	0.695	0.635
38	5	0.643	0.627	0.674	0.530
39	1	0.591	0.665	0.421	0.559
40	3	0.737	0.719	0.699	0.564
41	6	0.610	0.692	0.718	0.635
44	3	0.670	0.719	0.714	0.564
45	2	0.591	0.665	0.711	0.580
46	6	0.613	0.692	0.804	0.635
50	6	0.668	0.692	0.661	0.635
51	3	0.830	0.719	0.738	0.564
52	6	0.677	0.692	0.636	0.635
53	6	0.610	0.692	0.718	0.635

APPENDIX F : THE ALGORITHMS AND THE INTERNAL VARIABLES FOR FOUR THREATS

F.1. ALGORITHM 1

F.1.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w1=d1	w5=v1
w2=d2	w6=v2
w3=d3	w7=v3
w4=d4	w8=v4

w9=trunc(d1/10)=d11	w13=trunc(v1/10)=v11
w10=trunc(d2/10)=d21	w14=trunc(v2/10)=v21
w11=trunc(d3/10)=d31	w15=trunc(v3/10)=v31
w12=trunc(d4/10)=d41	w16=trunc(v4/10)=v41

```
w17     if d1<20 and v1>90 then Y=R1 stop,  else
w18     if d2<20 and v2>90 then Y=R2 stop
else
        w19=d11/v11=T1
        w20=d21/v21=T2

        w21= min(T1,T2)
        w22=distance of w21=d(w21)
        w23=speed of w21=v(w21)

w24     if d3<20 and v3>90 then Y=R3 stop
else
```

w25=trunc(d(w21)/10)
w26=trunc(v(w21)/10)
w27=w25 / w26=TS1
w28=d31/v31=T3

w29=min(TS1,T3)
w30=distance associated to w29, = d(w29)
w31=speed associated to w29, = v(w29)

w32 **if** d4<20 and v3>90 **then** Y=R1 **stop**
else

w33=trunc(d(w30/10))
w34=trunc(s(w31/10))
w35=w33/w34=TS2
w36=d41/v41=T4

w37=min(TS2,T4)
w38=ratio associated to w37=Y
stop

Output Vector Y

F.1.2 Explanatory Notes

The different notation used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables defined by the same notation are based on the same model.)

dij=jth digit of distance of ratio i. dij ranges from 1 to 9

w21= min(T1,T2). these variables may only take three values 0,1 or 2
0 if the two values are the same
1 if the first is smallest, T1 in this case
2 if the second is smallest, T2 in this case

w25,w29,w37 are the same type of internal variables w23 = distance associated to w22

w23 takes the value of the disance associated to the ratio corresponding to the value of w22. If w22 had taken a value of 1, w23 would take the values of d(R1), since R1 would be smaller than R2, such a ratio could be noted R1'. If w22 takes a value of 0, each ratio (either R1 or R2) has a probability of 0.5 of being chosen.

w26, w30 and w38 are the same type of internal variables.

F.2. ALGORITHM 2

F.2.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w1=d1	w5=v1
w2=d2	w6=v2
w3=d3	w7=v3
w4=d4	w8=v4

w9=trunc(d1/10)=d11	w13=trunc(v1/10)=v11
w10=trunc(d2/10)=d21	w14=trunc(v2/10)=v21
w11=trunc(d3/10)=d31	w15=trunc(v3/10)=v31
w12=trunc(d4/10)=d41	w16=trunc(v4/10)=v41

w17 **if** d1<20 and v1>90 **then** $Y=R_1$ **stop else**

w18 **if** d2<20 and v2>90 **then** $Y=R_2$ **stop else**

w19=round ((d1 - d11) /10)

w20=d11+w19

w21=round ((v1 - v11) /10)

w22=v11+w21

w23= round ((d2 - d21) /10)

w24 = d21+w23

w25=round ((v2 - v21)/10)

w26=v21+w26

w27=w22/w22=T_1

w28=w24w26=T_2

w29= min(T_1,T_2)

$w30$ =distance associated to ratio of $w29=d(R_{S1})$
$w31$ =speed associated to ratio of $w29=v(R_{S1})$

w32 **if** $d3<20$ and $v3>90$ **then** $Y=R_3$ **stop else**

$w33=\text{trunc}\,[d\,(R_{S1})/10)] = d1(R_{S1})$
$w34=\text{round}\,[\,(\,d(R_{S1}) - d1\,(R_{S1})\,)/10]$
$w35=d1(R_{S1})+w34$
$w36=\text{trunc}\,(v(R_{S1})/10) = v1(R_{S1})$
$w37=\text{round}\,[\,(\,v(R_{S1}) - v1(R_{S1})\,)/10]$
$w38=v1(R_{S1})+w37$
$w48=w35/w38=T_{S1}$
$w49=\text{round}\,[\,(\,d3 - d31\,)/10]$
$w50=d31+w49$
$w51=\text{round}\,[\,(v1 - v31)/10\,]$
$w52=v31+w51$
$w53=w51/w52= T_3$

$w54=\min(T_{S1},T_3)$
$w55$=ratio associated to $w54 = R_{S2}$

w56 **if** $d4<20$ and $v3>90$ **then** $Y=R_4$ **stop else**

$w57=\text{trunc}\,[d\,(R_{S2})/10)] = d1(R_{S2})$
$w58=\text{round}\,[\,(\,d(R_{S2}) - d1\,(R_{S2})\,)/10]$
$w59=d1(R_{S2})+w58$
$w60=\text{trunc}\,(v(R_{S2})/10) = v1(R_{S2})$
$w61=\text{round}\,[\,(\,v(R_{S2}) - v1(R_{S2})\,)/10]$
$w62=v1(R_{S1})+w61$
$w63=w59/w62=T_{S2}$
$w64=\text{round}\,[\,(\,d4 - d41\,)/10]$
$w65=d31+w64$
$w66=\text{round}\,[\,(v4 - v41)/10\,]$
$w67=v41+w66$
$w68=w65/w66= T_4$

$w69=\min(T_{S2},T_4)$
$w70$=ratio associated to $w69=Y$ **stop**

Output Vector Y

F.2.2 Explanatory Notes

The different notation used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables defined by the same notation are based on the same model.)

dij = jth digit of distance of ratio i. dij ranges from 1 to 9

$d1(R_{Si})$ = first digit of the distance of ratio (R_{Si}).
$d2(R_{Si})$ = second digit of the distance of ratio (R_{Si}).

$$w31 = \mathrm{round}(d12/10) = \begin{cases} 1 \text{ if } d12 \geq 5 \\ 0 \text{ if } d12 < 5 \end{cases}$$

$w41 = \min(T_1, T_2)$. these variables may only take three values 0,1 or 2
0 if the two values are the same
1 if the first is smallest, T_1 in this case
2 if the second is smallest, T_2 in this case

w41, w54, w67 are the same type of internal variables

w42 = ratio associated to w41
w42 takes the value of the ratio corresponding to the value of w41.
If w41 had taken a value of 1, w42 would take the values of R_1, given that R_1 is smaller than R_2, such a ratio could be noted R_{S1}. The probability distribution of R_{S1} is different from that of R1,(that of w25 is different than that of w1 or w2).
If w41 takes a value of 0, each ratio (either R_1 or R_2) has a probability of 0.5 of being chosen.

w42, w55 and w68 are the same type of internal variables.

$w44 = round(d2(R_{S1})/10)$ = first digit of the distance of the ratio corresponding to R_{S1}

$w46 = round(d_2(R_{S1})/10)$ = round off value of the 2nd digit of the distance of the ratio corresponding to R_{S1}.

F.3. ALGORITHM 3

F.3.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w_1= d1/v1=R1
w2=d2/v2=R2
w3=d3/v3=R3
w4=d4/v4=R4

w5 **if** [(d1/v1)<1] **then** continue page 3 for **ratios <1**

else ratios>1

w6=trunc(d1/v1)
w7=approximate(d1/v1-trunc(d1/v1)
w8=w14+w16=I_1
w9=trunc(d2/v2)
w10=approximate(d2/v2-trunc(d2/v2))
w11=w18+w20= I_2
w12 = min(I_1, I_2)
w13 = ratio associated to w12=R_{S1}

w14=trunc(d3/v3)
w15=approximate(d3/v3 - trunc(d3/v3))
w16=w14+w15=I_3
w17=trunc(R_{S1})
w18=approximate(R_{S1}- trunc(R_{S1}))

$w19 = w17 + w18 = I_{RS1}$
$w20 = \min(I_{RS1}, I_3)$

$w21$=ratio associated to $w20 = R_{S2}$

$w22 = trunc(d4/v4)$
$w23 = approximate(d4/v4 - trunc(d4/v4))$
$w24 = w22 + w23 = I_4$
$w25 = trunc(R_{S2})$
$w26 = approximate(R_{S2} - trunc(R_{S2}))$
$w27 = w25 + w26 = I_{RS2}$
$w28 = \min(I_{RS2}, I_4)$
$w29$=ratio associated to $w28 = Y$

ratios <1

w30 **if** $d1<20$ and $v1>90$ **then** $Y=R_1$ **stop**
else
w31 **if** $d2<20$ and $v2>90$ **then** $Y=R_2$ **stop**
else
$w32 = trunc(v1/d1)$
$w33 = approximate(v1/d1 - trunc(v1/d1))$
$w34 = w32 + w33 = I_1$
$w35 = trunc(v2/d2)$
$w36 = approximate(v2/d2 - trunc(d2/v2))$
$w37 = w35 + w36 = I_2$
$w38 = \max(I_1, I_2)$
$w39$ = ratio associated to $w38 = R_{S1}$

w40 **if** $d3<20$ and $v3>90$ **then** $Y=R_3$ **stop**
else
$w41 = trunc(v3/d3)$
$w42 = approximate(v3/d3 - trunc(v3/d3))$
$w43 = w41 + w42 = I_3$
$w44 = trunc(1/R_{S1})$
$w45 = approximate((1/R_{S1}) - trunc(1/R_{S1}))$

w46=w44+w45= I_{RS1}

w47 = max (I_{RS1}, I_3)

w48=ratio associated to w47=R_{S2}

w49 **if** d4<20 and v3>90 **then** Y=R_1 **stop**

else

w50=trunc(v4/d4)

w51=approximate(v4/d4 - trunc(v4/d4))

w52=w50+w51=I_4

w53=trunc(1/R_{S2})

w54=approximate((1/R_{S2}) - trunc(1/R_{S2}))

w55=w53+w54= I_{RS2}

w56=max(I_{RS2}, I_4)

w57=ratio associated to w56=Y

stop

Output Vector Y=di/vi

F.3.2 Explanatory Notes

The different notation used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables defined by the same notation are based on the same model.)

w12 = min(I_1, I_2). these variables may only take three values 0,1 or 2

0 if the two values are the same

1 if the first is smallest, I_1 in this case

2 if the second is smallest, I_2 in this case

w12, w20, w28 are the same type of internal variables

w38 = max(I_1, I_2). these variables may only take three values 0,1 or 2

$$w38 = \begin{cases} 0 \text{ if the two values are the same} \\ 1 \text{ if the first is largest, } I_1 \text{ in this case} \\ 2 \text{ if the second is largest, } I_2 \text{ in this case} \end{cases}$$

w38,w47,w56 are the same type of internal variables

w13 = ratio associated to w12

w13 takes the value of the ratio corresponding to the value of w12.
If w12 had taken a value of 1, w13 would take the values of R_1, given that R_1 is smaller than R_2, such a ratio could be noted R_1'.
If w12 takes a value of 0, each ratio (either R_1 or R_2) has a probability of 0.5 of being chosen.

w13, w21, w29, w39,w48,w57 are the same type of internal variables.

w7 = approximate(d1/v1-trunc(d1/v1)

w7 may only take 3 values 0, 0.25 or 0.75.

$$w16 = \begin{cases} 0.00 \text{ if approximate(..)} < 0.25 \\ 0.75 \text{ if approximate(..)} > 0.75 \\ 0.25 \text{ otherwise} \end{cases}$$

w7, w10, w15, w18, w23, w33, w36, w42, w45, w48, w51, w54 are of the same type.

F.4. ALGORITHM 4

F.4.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w1=d1	w5=v1
w2=d2	w6=v2
w3=d3	w7=v3
w4=d4	w8=v4

w9 **if** d1<20 and v1>90 **then** $Y=R_{1,}$ **stop** **else**

w10 **if** d2<20 and v2>90 **then** $Y=R_{2,}$ **stop** **else**

w11 = min(d1,d2)
w12= max(v1,v2)
w13 = corresp(w11,w12)

w14= d1-v1+10
w15 = d2-v2
w16=min(w14,w15)
w17=d2-v2+10
w18=d1-v1
w19=min(w17,w18)
w20=distance of ratio associated to(w13,w16,w19)=d (R_{S1})
w21=distance of ratio associated to(w13,w16,w19)=v (R_{S1})

w22 **if** d3<20 and v3>90 **then** Y=R3 **stop**
else
w23= min(d (R_{S1}),d3)

$w24 = \max(v(R_{S1}),v3)$
$w25 = corresp(w23,w24)$
$w26 = d(R_{S1}) - v(R_{S1}) + 10$
$w27 = d4 - v4$
$w28 = \min(w26,w27)$
$w29 = d4 - v4 + 10$
$w30 = d(R_{S1}) - v(R_{S1})$
$w31 = \min(w28, w29)$

w32=distance associated to the ratio (w25,w28,w31) = $d(R_{S2})$
w33=distance associated to the ratio (w25,w28,w31) = $v(R_{S2})$

w34 **if** $d4<20$ and $v4>90$ **then** $Y=R_4$ **stop**
else
$w35 = \min(d(R_{S2}),d4)$
$w36 = \max(v(R_{S2}),v4)$
$w37 = corresp(w53,w54)$
$w38 = d(R_{S2}) - v(R_{S2}) + 10$
$w39 = d4 - v4$
$w40 = \min(w38,w39)$
$w41 = d(R_{S2}) - v(R_{S2})$
$w42 = d4 - v4 + 10$
$w43 = \min(w41,w42)$
w44 = ratio associated to (w37,w40,w43) = Y = di/vi

Output Vector Y

F.4.2 Explanatory Notes

The different notation used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables described using the same notation are based on the same model.)

$$\text{w11} = \min(\text{d1, d2}) = \begin{cases} \text{1 if the first element (in this case d1) is the smallest} \\ \text{2 if the second element (in this case d2) is the smallest} \end{cases}$$

$$\text{w12} = \max(\text{v1, v2}) = \begin{cases} \text{1 if the first element (in this case v1) is the largest} \\ \text{2 if the second element (in this case vd2) is the largest} \end{cases}$$

$$\text{w13} = \text{corresp (w15,w16)} = \begin{cases} \text{1 if } \text{w15} = \text{w16} = 1 \\ \text{2 if } \text{w15} = \text{w16} = 2 \\ \text{0 otherwise} \end{cases}$$

$$\text{w14} = \text{d1-v1} + 10 = \begin{cases} \text{d1-v1+10 if } \text{w13} = 0 \\ \text{nonexistent otherwise} \end{cases}$$

$$\text{w15} = \text{d2} - \text{v2} = \begin{cases} \text{d2-v2 if } \text{w13} = 0 \\ \text{nonexistent otherwise} \end{cases}$$

$$\text{w16} = \min(\text{w14, w15}) = \begin{cases} \text{1 if w13=0 and w14} < \text{w15} \\ \text{nonexitent if w13} <> 0 \\ \text{0 otherwise} \end{cases}$$

$$\text{w17} = \text{d2} - \text{v2} + 10 = \begin{cases} \text{d2-v2+10 if } \text{w13} = 0 \text{ and } \text{w16} = 0 \\ \text{nonexistent otherwise} \end{cases}$$

$$\text{w18} = \text{d2} - \text{v2} = \begin{cases} \text{d2-v2 if } \text{w13} = 0 \text{ and } \text{w16} = 0 \\ \text{nonexistent otherwise} \end{cases}$$

$$\text{w19} = \min(\text{w17, w18}) = \begin{cases} \text{2 if } \text{w13} = 0, \text{w16} = 0, \text{ and } \text{w17} < \text{w18} \\ \text{0 if } \text{w13} = 0 \text{ and } \text{w16} = 0 \text{ and } \\ \qquad (\text{w17} > \text{w18 or w17} = \text{w18}) \\ \text{nonexistent otherwise} \end{cases}$$

$$
w20 = \text{ratio associated to } (w13, w16, w19) = \begin{cases} R_1 \text{ if } w13 = 1 \text{ or } w16 = 1 \\ R_2 \text{ if } w13 = 2 \text{ or } w16 = 2 \\ R_1 \text{ or } R_2 \ (p = 0.5) \text{ if } w19 = 0 \end{cases}
$$

F.5. ALGORITHM 5

F.5.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w1=d1	w5=v1
w2=d2	w6=v2
w3=d3	w7=v3
w4=d4	w8=v4

w9 **if** d1<20 and v1>90 **then** $Y=R_1$ **stop else**

w10 **if** d2<20 and v2>90 **then** $Y=R_2$ **stop else**

w11= min(d1,d2)
w12 = max(v1,v2)
w13 = corresp(w11,w12)

w14 = d1-d2
w15 = v1-v2
w16 =min(w14,w15)
w17 = distance of ratio associated to (w13,w16) = d (R_{S1})
w18 = speed of ratio associated to (w13,w16) = v (R_{S1})

w19 **if** d3<20 and v3>90 **then** $Y=R_3$ **stop else**

w20= min(d (R_{S1}),d3)
w21 = max(v (R_{S1}),v3)
w22 = corresp(w20,w21)

$w23 = d(R_{S1}) - d3$

$w24 = v(R_{S1}) - v3$

$w25 = \min(w23, w24)$

$w26$ = distance of ratio associated to $(w36, w29) = d(R_{S2})$

$w26$ = speed of ratio associated to $(w36, w29) = v(R_{S2})$

w27 **if** $d4<20$ **and** $v4>90$ **then** $Y=R_4$ **stop else**

$w28 = \min(d(R_{S2}), d4)$

$w29 = \max(v(R_{S2}), v4)$

$w30 = \text{corresp}(w39, w40)$

$w31 = d(R_{S2}) - d4$

$w32 = v(R_{S2}) - v4$

$w33 = \min(w31, w32)$

$w49$ = ratio associated to$(w30, w33) = Y$

Output Vector Y

F.5.2 Explanatory Notes

The different notation used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables described using the same notation are based on the same model.)

$$w11 = \min(d1, d2) = \begin{cases} 1 \text{ if the first element (in this case d1) is the smallest} \\ 2 \text{ if the second element (in this case d2) is the smallest} \end{cases}$$

$$w12 = \max(v1, v2) = \begin{cases} 1 \text{ if the first element (in this case v1) is the largest} \\ 2 \text{ if the second element (in this case v2) is the largest} \end{cases}$$

$$w13 = \text{corresp}\ (w11, w12) = \begin{cases} 1 \text{ if } w11 = w12 = 1 \\ 2 \text{ if } w11 = w12 = 2 \\ 0 \text{ otherwise} \end{cases}$$

$$w14 = d1 - d2 = \begin{cases} d1 - d2 \text{ if } w13 = 0 \\ \text{nonexistent otherwise} \end{cases}$$

$$w16 = \min\ (w14, w15) = \begin{cases} 1 \text{ if } w13 = 0 \text{ and } w14 < w15 \\ 2 \text{ if } w13 = 0 \text{ and } w14 > w15 \\ \text{nonexitent if } w13 <> 0 \\ 0 \text{ otherwise} \end{cases}$$

$$w17 = \text{ratio associated to } (w13, w16) = \begin{cases} R_1 \text{ if } w13 = 1 \text{ or } w16 = 1 \\ R_2 \text{ if } w13 = 2 \text{ or } w16 = 2 \\ R_1 \text{ or } R_2\ (p = 0.5) \text{ if } w19 = 0 \end{cases}$$

F.6. ALGORITHM 6

F.6.1 Definition of Variables

Input vector X

X=(d1/v1, d2/v2, d3/v3, d4/v4)

Internal Variables

w1=d1	w5=v1
w2=d2	w6=v2
w3=d3	w7=v3
w4=d4	w8=v4

w9 = trunc(d1/10) = d11	w13 =trunc(v1/10) = v11
w10 =trunc(d2/10) = d21	w14 =trunc(v2/10) = v21
w11 =trunc(d3/10) = d31	w15 =trunc(v3/10) = v31
w12 =trunc(d4/10) = d41	w16 =trunc(v4/10) = v41

w17 **if** d1<20 and v1>90 **then** $Y=R_1$ **stop else**

w18 **if** d2<20 and v2>90 **then** $Y=R_2$ **stop else**

w19 = min(d1,d2)
w20 = max(v1,v2)
w21 = corresp(w15,w16)

w22 = round [(d1 - d11)/10]
w23 = d11+w22
w24 = round [(v1 - v11)/10]
w25 = v11+w24
w26 = w22/w23 = T_1
w25 = round [(d2 - d21)/10]
w26 = d21+w26

$w27 = \text{round}\,[(v2 - v21)/10]$

$w28 = v21 + w27$

$w29 = w26/w28 = T_2$

$w30 = \min(w26, w29)$

w31= distance of the ratio associated to(w21,w30) = d (R_{S1})

w32= distance of the ratio associated to(w21,w30) = v (R_{S1})

w33 **if** d3<20 and v3>90 **then** Y=R_3 **stop else**

$w34 = \min(d(R_{S1}), d3)$

$w35 = \max(v(R_{S1}), v3)$

$w36 = \text{corresp}(w34, w35)$

$w37 = \text{trunc}\,(d(R_{S1})/10) = d1(R_{S1})$

$w38 = \text{round}\,[(d(R_{S1}) - d1(R_{S1})/10]$

$w39 = d1(R_{S1}) + w38$

$w40 = \text{trunc}\,(v(R_{S1})/10) = v1(R_{S1})$

$w41 = \text{round}\,[(v(R_{S1}) - v1(R_{S1})/10]$

$w42 = v1(R_{S1}) + w41$

$w43 = w39 / w42 = T_{RS1}$

$w44 = \text{round}\,[(d3 - d31)/10]$

$w45 = d31 + w27$

$w46 = \text{round}\,[(v3 - v31)/10]$

$w47 = v31 + w46$

$w48 = w45/w47 = T_3$

$w49 = \min(T_{RS1}, T_3)$

w50 = distance of ratio associated to(w36,w49) =d (R_{S2})

w51 = distance of ratio associated to(w36,w49) =v (R_{S2})

w52 **if** d4<20 and v4>90 **then** Y=R_4 **stop else**

$w53 = \min(d(R_{S2}),d4)$
$w54 = \max(v(R_{S2}),v4)$
$w55 = \text{corresp}\ (w47,w48)$

$w56 = \text{trunc}\ (\ d(R_{S2})/10\ ) = d1(R_{S2})$
$w57 = \text{round}\ [(\ d(R_{S2}) - d1(R_{S2})\ /10]$
$w58 = d1(R_{S2}) + w57$
$w59 = \text{trunc}\ (\ v(R_{S2})/10\ ) = v1(R_{S2})$
$w60 = \text{round}\ [(\ v(R_{S2}) - v1(R_{S2})\ /10]$
$w61 = v1(R_{S2}) + w60$
$w62 = w58\ /\ w61 = T_{RS2}$
$w63 = \text{round}\ [\ (d4 - d41)\ /\ 10]$
$w64 = d41 + w63$
$w65 = \text{round}\ [\ (v4 - v41)\ /\ 10]$
$w66 = v41 + w65$
$w67 = w64/w66 = T_4$
$w68 = \min(\ T_{RS2}, T_4\ )$

w69=ratio associated to(w55,w68) =Y

Output Vector Y

F.6.2 Explanatory Notes

The different notations used in the previous algorithm may be described as follows: (Only one variable of each type is described. The other variables defined by the same notation are based on the same model.)

$$w19 = \min(d1, d2) = \begin{cases} 1 \text{ if the first element (in this case d1) is the smallest} \\ 2 \text{ if the second element (in this case d2) is the smallest} \end{cases}$$

$$w20 = \max(v1, v2) = \begin{cases} 1 \text{ if the first element (in this case v1) is the largest} \\ 2 \text{ if the second element (in this case v2) is the largest} \end{cases}$$

$$w21 = \text{corresp}\ (w15, w16) = \begin{cases} 1 \text{ if } w15 = w16 = 1 \\ 2 \text{ if } w151 = w16 = 2 \\ 0 \text{ otherwise} \end{cases}$$

If w21 is not equal to 0 , the variables w21 to w29 have a probability of 0 of occuring, that is they only exist in the case when w21 is equal to 0.

$$w17 = \text{distance of ratio associated to } (w17, w28) = \begin{cases} R_1 \text{ if } w21 = 1 \text{ or } w30 = 1 \\ R_2 \text{ if } w21 = 2 \text{ or } w30 = 2 \\ d_1 \text{ or } d_2\ (p = 0.5) \text{ if } w21 = 0 \text{ and } \\ \quad w30 = 0 \end{cases}$$

APPENDIX G

ENTROPY OF THE VARIABLES OF THE ALGORITHMS

The entropy of most variables described in Chapter VI was derived using computer simulations. In some cases however, the entropy of 'non-static variables' was estimated from that of other similar variables: The decrease in entropy after each stage was approximated to that of a similar variable. For each algorithm the approximations were different.

G.1 ALGORITHM 1

For Algorithm 1, the decrease in entropy of the truncated speeds and distances were derived from the rate of decrease of the speeds and distances after each decision. The rate is noted 'Multiplyer' in Table G.1

The variables of Table G.1 have been defined in Appendix F. They are described again for Algorithm 1, but will not be described later.

di: distance of ratio i
vi: speed of ratio i
dsi: distance of ratio chosen as smallest at decision si-1, where si =2 to 6
vsi: speed of ratio chosen as smallest at decision si-1
di1: first digit of the distance of ratio i
vi1: first digit of the speed of ratio i
trunc dsi: first digit of the distance of ratio chosen as smallest at decision si-1
trunc vsi: first digit of the speed of ratio chosen as smallest at decision si-1
decision i: decision variable to choose the smallest of the two ratios at comparison i
RSi = trunc dsi / trunc vsi: Variable used to make comparison si
di < 20 and vi > 90: Decision variable to check for very small ratios

Table G.1 Algorithm 1

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six tasks
input				45.764	77.683
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
di1		4 or 7	3.165	12.661	22.156
vi1		4 or 7	3.158	12.630	22.103
di<20 and vi>90		4 or 7	0.183	0.733	1.282
di1/vi1		4 or 7	6.248	24.992	43.735
decision1			1.150	1.150	1.150
decision 2			1.082	1.082	1.082
ds2			6.229	6.229	6.229
vs2			6.244	6.244	6.244
trunc ds2	1.029		3.076	3.076	3.076
trunc vs2	1.026		3.077	3.077	3.077
Rs2			5.654	5.654	5.654
ds3			6.056	6.056	6.056
vs3			6.021	6.021	6.021
decision3			1.004	1.004	1.004
trunc ds3	1.029		2.991	2.991	2.991
trunc dv3	1.037		2.967	2.967	2.967
Rs3			5.457	5.457	5.457
ds4			5.924		5.924
vs4			5.761		5.761
decision4			0.952		0.952
trunc ds4	1.022		2.926		2.926
trunc dv4	1.045		2.839		2.839
Rs4			5.126		5.126
ds5			5.821		5.821
vs5			5.604		5.604
decision5			0.920		0.920
trunc ds5	1.018		2.875		2.875
trunc vs5	1.028		2.761		2.761
Rs5			4.852		4.852
ds6			5.736		5.736
vs6			5.422		5.422
decision 6			0.902		0.902
trunc ds6	1.015		2.833		2.833
trunc vs6	1.034		2.671		2.671
Rs6			4.619		4.619
Output				11.042	10.460
Total Entropy				210.103	386.700

G.2 ALGORITHM 2

For Algorithm 2, the decrease in entropy of the rounded speeds and distances were derived from the rate of decrease of the speeds and distances after each decision. The rate is noted 'Multiplyer' in Table G.2. The variables of Table G.2 are very similar to those of Table G.1, except that 'round dsi' is used instead of 'trunc dsi'. Also round(di2/10) is the approximated value of the second digit when it is divided by 10. It can only take a value of 0 or 1.

Table G.2 Algorithm 2

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
				45.764	77.683
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
di<20 and vi>90		4 or 7	0.183	0.734	1.284
di1		4 or 7	3.165	12.661	22.156
vi1		4 or 7	3.158	12.630	22.103
round(di2/10)		4 or 7	0.993	3.973	6.952
round(vi2/10		4 or 7	0.993	3.973	6.952
round di		4 or 7	3.239	12.956	22.673
round vi		4 or 7	3.249	12.996	22.742
rou(di)/rou(vi)		4 or 7	6.428	25.711	44.994
decision1			1.145	1.145	1.145
di			6.169	6.169	6.169
vi			6.316	6.316	6.316
di2			3.047	3.047	3.047
vi2			3.112	3.112	3.112
round(di2/10)			0.956	0.956	0.956
round(vi2/10			0.979	0.979	0.979
round ds2	1.039		3.127	3.127	3.127
round vs2	1.015		3.202	3.202	3.202
Rs2			6.050	6.050	6.050
decision 2			1.076	1.076	1.076
di			5.996	5.996	5.996
vi			6.167	6.167	6.167
di3			2.962	2.962	2.962
vi3			3.039	3.039	3.039
round(di3/10)			0.929	0.929	0.929
round(vi3/10			0.956	0.956	0.956
round ds3	1.029		3.040	3.040	3.040

Table G.2. (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
round vs3	1.024		3.126	3.126	3.126
Rs3			5.933	5.933	5.933
decision3			0.995	0.995	0.995
di			5.870		5.870
vi			6.020		6.020
di4			2.899		2.899
vi4			2.966		2.966
round(di4/10)			0.910		0.910
round(vi4/10			0.933		0.933
round ds4	1.021		2.976		2.976
round dv4	1.024		3.052		3.052
Rs4			5.812		5.812
decision4			0.940		0.940
di			5.773		5.773
vi			5.883		5.883
di5			2.851		2.851
vi5			2.899		2.899
round(di5/10)			0.895		0.895
round(vi5/10			0.912		0.912
round ds5	1.017		2.926		2.926
round dv5	1.023		2.983		2.983
Rs5			5.347		5.347
decision5			0.905		0.905
di			5.698		5.698
vi			5.757		5.757
di6			2.814		2.814
vi6			2.836		2.836
round(di6/10)			0.883		0.883
round(vi6/10			0.892		0.892
round ds6	1.013		2.889		2.889
round vs6	1.022		2.918		2.918
Rs6			5.151		5.151
decision 6			0.884		0.884
Output				11.042	8.394
Total Entropy				262.031	480.059

G.3 ALGORITHM 3

Algorithm 3 considers the ratios less than one and the ratios larger than one separately. The multipliers were only required for the ratios larger than one. (Many of the variables have already been defined for Table G.1 in section G.1.1.) The following variables still need to be defined:

integer = trunc(di/vi)

round leftover = round [di/vi - trunc(di/vi)

ratio RS1 = integer + round, the value used to make the comparison

Table G.3 Algorithm 3

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
Input				45.764	77.683
di/vi		4 or 7	12.393	49.570	86.748
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
Ratios >1		0.471	0.520	0.991	0.991
integer		4 or 7	1.968	7.873	13.777
round leftover		4 or 7	1.357	5.430	9.502
ratio RS1		4 or 7	2.264	9.054	15.845
decision1			1.491	1.491	1.491
di/vi			10.955	10.955	10.955
round left over			1.291	1.291	1.291
integer			1.087	1.087	1.087
ratio RS2			1.215	1.215	1.215
decision2			1.494	1.494	1.494
di/vi			10.427	10.427	10.427
round left over			1.248	1.248	1.248
integer			0.736	0.736	0.736
ratio RS3			1.712	1.712	1.712
decision3			1.491	1.491	1.491
di/vi			10.173		10.173
round left over			1.210		1.210
integer			0.599		0.599
ratio RS4			1.444		1.444
decision4			1.487		1.487

Table G.3 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
di/vi			9.948		9.948
round left over			1.167		1.167
integer			0.552		0.552
ratio RS5			1.285		1.285
decision5			1.459		1.459
di/vi			9.763		9.763
round left over			1.115		1.115
integer			0.539		0.539
ratio RS6			1.173		1.173
decision6			1.409		1.409
Ratios<1					
di<20 and vi>90			0.139	0.558	0.976
trunc(v1/d1)		4 or 7	1.647	6.587	11.528
appr. left. trunc		4 or 7	1.357	5.430	9.502
trunc(vi/di)+app		4 or 7	2.264	9.054	15.845
decision1			1.484	1.484	1.484
di/vi			10.747	10.747	10.747
round left over	0.915		1.484	1.484	1.484
integer			2.281	2.281	2.281
ratio RS2	0.634		3.571	3.571	3.571
decision2			1.301	1.301	1.301
di/vi			10.280	10.280	10.280
round left over	1.017		1.459	1.459	1.459
integer			2.405	2.405	2.405
ratio RS3	0.984		3.631	3.631	3.631
decision3			1.169	1.169	1.169
di/vi			9.968		9.968
round left over	0.999		1.460		1.460
integer			2.421		2.421
ratio RS4	1.003		3.619		3.619
decision4			1.094		1.094
di/vi			9.730		9.730
round left over	1.004		1.454		1.454
integer			2.409		2.409
ratio RS5	1.009		3.586		3.586
decision5			1.050		1.050
di/vi			9.534		9.534

Table G.3 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
round left over	1.005		1.446		1.446
integer			2.389		2.389
ratio RS6	1.010		3.551		3.551
decision6			1.024		1.024
Output				11.042	10.460
Total				275.582	513.594

G.4 ALGORITHM 4

The change in entropy after each comparison has been set to be the same as for Algorithm 5, since the two algorithms are very similar.

Table G.4. Algorithm 4

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
input				45.764	77.683
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
di<20,vi>90		4 or 7	0.139	0.558	0.976
Decision 1					
min(d1,d2)			3 or 6	1.000	1.000
max(v1,v2)			3 or 6	1.000	1.000
corr	used al5			1.220	1.220
di-vi+10			4 or 7	6.911	6.911
di-vi				6.886	6.886
min1				0.988	0.988
di-vi+10				6.537	6.537
di-vi				6.829	6.829
min2				0.881	0.881
Decision2					
di			6.321	6.321	6.321
vi			6.321	6.321	6.321
min(d(Rs1),d3)			0.925	0.925	0.925
max(v(RS1),v3)			0.925	0.925	0.925
corr			1.128	1.128	1.128
di-vi+10	used al5		7.007	7.007	7.007
di-vi			6.791	6.791	6.791
min1			0.914	0.914	0.914
di-vi+10			6.447	6.447	6.447
di-vi			6.735	6.735	6.735
min2			0.815	0.815	0.815
Decision3					
di			6.073	6.073	6.073
vi			6.073	6.073	6.073
min(d(Rs2) ,d4)			0.833	0.833	0.833
max(v(RS1),v4)			0.833	0.833	0.833
corr			1.016	1.016	1.016
di-vi+10			6.733	6.733	6.733

Table G.4 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
di-vi			6.525	6.525	6.525
min1			0.823	0.823	0.823
di-vi+10			6.358	6.358	6.358
di-vi			6.643	6.643	6.643
min2			0.734	0.734	0.734
Decision 4					
di			5.875		5.875
vi			5.875		5.875
min(d(Rs3) ,d5)			0.761		0.761
max(v(Rs3),v5)			0.761		0.761
corr			0.929		0.929
di-vi+10			6.513		6.513
di-vi			6.312		6.312
min1			0.752		0.752
di-vi+10			6.150		6.150
di-vi			6.425		6.425
min2			0.671		0.671
Decision 5					
di			5.710		5.710
vi			5.710		5.710
min(d(Rs4),d6)			0.709		0.709
max(v(Rs4),v6)			0.709		0.709
corr			0.864		0.864
di-vi+10			6.331		6.331
di-vi			6.135		6.135
min1			0.700		0.700
di-vi+10			5.978		5.978
di-vi			6.246		6.246
min2			0.624		0.624
Decision 6					
di			5.573		5.573
vi			5.573		5.573
min(d(Rs5) ,d7)			0.670		0.670
max(v(Rs5),v7)			0.670		0.670
corr			0.817		0.817
di-vi+10			6.178		6.178
di-vi			5.987		5.987
min1			0.661		0.661
di-vi+10			5.834		5.834
di-vi			6.095		6.095
min2			0.590		0.590
Output entropy				11.042	10.460
Total entropy				227.858	417.450

G.5 ALGORITHM 5

The rate of change of entropy of the variables of algorithm 5 were assumed to be the same as taht of the the decision variable. The variables were described in Appendix F.

Table G.5. Algorithm 5

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
input				45.764	77.683
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
di<20,vi>90		4 or 7	0.087	0.347	0.607
min(d1,d2)	decrease		1.000	1.000	1.000
max(v1,v2)	same as the		1.000	1.000	1.000
CORR	decisions 1		1.228	1.228	1.228
di-vi			6.975	6.975	6.975
dec1			1.100	1.100	1.100
de2	1.081		1.018	1.018	1.018
di			6.321	6.321	6.321
vi			6.321	6.321	6.321
di-vi	1.014		6.879	6.879	6.879
min(d1,d2)			0.925	0.925	0.925
max(v1,v2)			0.925	0.925	0.925
CORR			1.136	1.136	1.136
de3	1.111		0.916	0.916	0.916
di			6.073	6.073	6.073
vi			6.073	6.073	6.073
di-vi	1.041		6.610	6.610	6.610
min(d1,d2)			0.833	0.833	0.833
max(v1,v2)			0.833	0.833	0.833
CORR			1.023	1.023	1.023
de4	1.094		0.838		0.838
di			5.875		5.875
vi			5.875		5.875
di-vi	1.034		6.394		6.394
min(d1,d2)			0.761		0.761
max(v1,v2)			0.761		0.761
CORR			0.935		0.935
					0.000
de5	1.075		0.780		0.780
di			5.710		5.710
vi			5.710		5.710

Table G.5 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
di-vi	1.029		6.215		6.215
min(d1,d2)			0.709		0.709
max(v1,v2)			0.709		0.709
CORR			0.870		0.870
de6	1.058		0.737		0.737
di			5.573		5.573
vi			5.573		5.573
di-vi	1.025		6.065		6.065
min(d1,d2)			0.670		0.670
max(v1,v2)			0.670		0.670
CORR			0.823		0.823
Output				11.042	10.460
Total Entropy				165.615	297.915

G.6 ALGORITHM 6

The rate at which the entropy changes after each decision for the non-static variables of algorithm 6 is derived from algorithm 5 and algorithm 2, since this algorithm (6) is a combination of both.

Table G.6 Algorithm 6

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
input				45.764	77.683
di		4 or 7	6.409	25.635	44.862
vi		4 or 7	6.409	25.635	44.862
di<20 and vi>90		4 or 7	0.087	0.347	0.607
di1		4 or 7	3.165	13.168	23.045
vi1		4 or 7	3.158	12.672	22.176
min(d1,d2)			1.000	1.000	1.000
max(v1,v2)			1.000	1.000	1.000
CORR			0.549	0.549	0.549
round(di2/10)			0.994	3.974	6.955
round(vi2/10			0.994	3.974	6.955
round di			3.238	12.950	22.663
round vi			3.243	12.971	22.699
Rs1			6.424	25.695	44.966
dec1			1.238	1.238	1.238
di			6.169	6.169	6.169
vi			6.316	6.316	6.316
di1			3.047	3.047	3.047
vi1			3.112	3.112	3.112
min(ds2,di)			0.927	0.927	0.927
max(vs2,vi)			0.927	0.927	0.927
CORR			0.508	0.508	0.508
round(di2/10)			0.956	0.956	0.956
round(vi2/10			0.979	0.979	0.979
round ds2			3.116	3.116	3.116
round vs2			3.196	3.196	3.196
RS2			6.046	6.046	6.046
dec2	1.079		1.148	1.148	1.148
di			5.996	5.996	5.996
vi			6.167	6.167	6.167
di1			2.962	2.962	2.962
vi1			3.039	3.039	3.039
min(ds3,di)			0.830	0.830	0.830
max(vs3,vi)			0.830	0.830	0.830
CORR			0.455	0.455	0.455

Table G.6 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
round(di2/10)			0.930	0.930	0.930
round(vi2/10			0.956	0.956	0.956
round ds3		3.029	3.029	3.029	
round vs3			3.120	3.120	3.120
Rs3			5.592	5.592	5.592
dec3	1.117		1.028	1.028	1.028
di			5.870		5.870
vi			6.020		6.020
di1			2.899		2.899
vi1			2.966		2.966
min(ds4,di)			0.804		0.804
max(vs4,vi)			0.804		0.804
CORR			0.441		0.441
round(di2/10)			0.936		0.936
round(vi2/10			0.933		0.933
round ds4			2.965		2.965
round vs4			3.046		3.046
Rs4			5.478		5.478
decision4	1.033		0.995		0.995
di			5.773		5.773
vi			5.883		5.883
di1			2.851		2.851
vi1			2.899		2.899
min(ds5,di)			0.731		0.731
max(vs5,vi)			0.731		0.731
CORR			0.401		0.401
round(di2/10)			0.920		0.920
round(vi2/10			0.912		0.912
round ds5			2.916		2.916
round vs5			2.977		2.977
Rs5			5.040		5.040
decision5	1.100		0.905		0.905
di			5.698		5.698
vi			5.757		5.757
d1			2.814		2.814
v1			2.836		2.836
min(ds6,di)			0.714		0.714
max(vs6,vi)			0.714		0.714
CORR			0.392		0.392
round(di2/10)			0.909		0.909
round(vi2/10			0.892		0.892

Table G.6 (Continued)

Description of Variable	Multiplyer	Frequency	Entropy	Trials of Three Tasks	Trials of Six Tasks
round ds6			2.879		2.879
round vs6			2.913		2.913
RS6			4.932		4.932
decision6	1.024		0.884		0.884
Output				11.042	10.460
Total Entropy				268.995	502.530

END

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